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Synthesis of robust memory modes for linear quantum systems with unknown inputs

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Abstract

In this paper, the synthesis of robust memory modes for linear quantum passive systems in the presence of unknown inputs has been studied, aimed at facilitating secure storage and communication of quantum information. In particular, we can switch on decoherence-free (DF) modes in the storage stage by placing the poles on the imaginary axis via a coherent feedback control scheme, and these memory modes can further be simultaneously made robust against perturbations to the system parameters by minimizing the condition number associated with imaginary poles. The DF modes can also be switched off by tuning the controller parameters to place the poles in the left half of the complex plane in the writing/reading stage. We develop explicit algebraic conditions guiding the design of such a coherent quantum controller, which involves employing an augmented system model to counter the influence of unknown inputs. Examples are provided to illustrate the procedure of synthesizing robust memory modes for linear optical quantum systems.

Keywords: Linear quantum passive systems; Unknown inputs; Robust memory modes

1 Introduction

Quantum communication networks, described by quantum stochastic differential equations and the associated (S, L, H) framework, contain environmental noise whose intensity is sufficiently high in reality. Due to the existence of environmental noise, open quantum systems suffer from loss of coherence. In more concrete terms, the degradation of the superposition of distinct quantum states into a classical mixture under the action of the environment is often called decoherence, which is detrimental to the processing and storage of quantum information [1, 2]. Therefore, it is important to develop tools for the engineering of decoherence-free (DF) modes [3–10], aimed at improving the resilience of quantum communication systems to environmental and channel distortions. When being put in a DF mode which is immune to noise, the quantum information associated with the target quantum state can be preserved, making it an ideal candidate for the implementation of quantum memory [11]. Moreover, quantum computation can also be protected against decoherence if implemented within a DF mode [12].

It is thus of much importance to consider the writing, holding (storage) and reading stages of quantum information concerning reliable quantum communication and computation at a relatively abstract level. First of all, the target quantum system can be made to contain a memory mode that can store quantum states without loss over a long period of time. Ideally, this mode should be completely decoupled from other system components and the surrounding environment while it stores the state; i.e., it is a DF mode. In particular, in the storage stage, this DF mode is decoupled even from the channel used for transferring an input state or retrieving the stored state. Secondly, during the writing/reading stage, the system should be tuned so that the memory mode couples to the associated transportation channel.

On the one hand, memory qubits can be used to faithfully preserve quantum coherence and correlations where control pulses may be involved to improve the memory's robustness [13–16]. On the other hand, optimal control for perfect state transfer in linear quantum memory has also been considered in order to directly take advantage of DF modes [17]. Apart from the environment dissipation, uncertainties inevitably exist in open quantum systems [18–21], and therefore we focus on the synthesis of robust memory modes in the presence of unknown inputs, using only passive optical components like phase shifters, beam splitters and mirrors, by means of coherent feedback control in this paper. In fact, linear quantum passive systems are relevant to information processing of various systems such as optical cavities and nano-mechanical oscillators, for which a systematic theory has been derived to model the dynamics of the corresponding quantum operators, or quantum modes, in the Heisenberg picture [22, 23]. In particular, linear quantum systems can usually be interconnected to create a network by using optical fields as inputs and outputs [24–26]. If a quantum mode is directly or indirectly coupled to the input field, then this mode can be affected by external noise. Additionally, if a quantum mode is coupled to the output field, then the information stored in this mode can leak out [6, 8, 27, 28]. Consequently, in this case only quantum states governed by the DF modes as mentioned above could be protected from decoherence.

The system setup concerning the synthesis of robust memory modes is demonstrated in Fig. 1. The original plant may not have DF modes, as each one of its modes may be coupled to the input and thus not isolated from the environment. To create DF modes, another linear quantum system, playing the role of a Luenberger-type coherent controller,

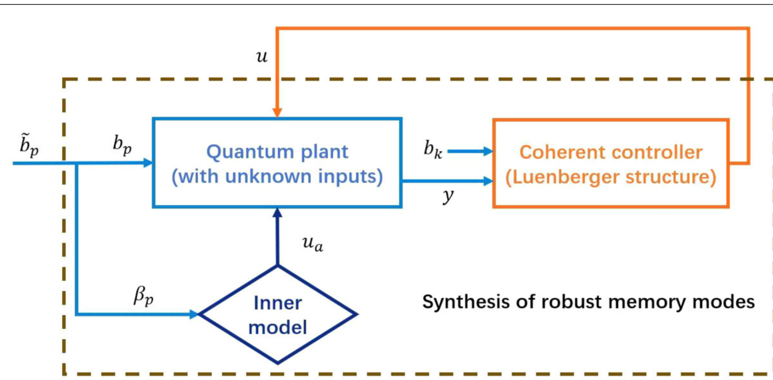

Figure 1 Coherent feedback design for the synthesis of robust memory modes in the presence of unknown inputs (β_p included in $\tilde{b}_p$). The Luenberger-type coherent controller is realized by another linear quantum system, which is interconnected with the original plant via optical fields as the input and output (e.g. y and u). An inner model associated with the auxiliary variable (u_a) is employed to handle unknown inputs in the plant

is introduced such that DF modes corresponding to tunable memory modes can be constructed in the augmented system. An auxiliary inner model will be employed in order to handle unknown inputs in the plant which characterize uncertainties in open quantum systems. More precisely, by tuning controller parameters, we can switch between opening and closing of DF modes; namely the memory modes are DF during the storage period while they are not DF in the writing/reading process. We are particularly interested in linear quantum passive systems where energy generation is not involved. It has been proven in [29] that every purely imaginary pole of a linear quantum passive system indicates the existence of a DF mode that is isolated from the inputs and outputs, thereby turning the original synthesis problem into a pole placement problem.

Pole placement within the scope of coherent feedback as depicted in Fig. 1 has been studied in our previous work, under a coherent observer-based framework [30, 31]. Although the pole placement problem has been tackled in [30] by solving an algebraic equation, robust pole placement applied to the synthesis of memory modes in the presence of unknown inputs is not yet considered [21]. For practical quantum control systems, it is critical to attain robustness due to prevalent uncertainties in open quantum systems [32–34]. Considering the linear quantum passive system, the poles of such a system can be made either purely imaginary or in the open left-half complex plane. However, any perturbation involved in the system coefficient matrices may move the poles away from the imaginary axis, causing the memory mode to decay information to the external environment. For this reason, the sensitivity of desired imaginary poles to perturbations should be minimized. There exist a variety of eigenstructure assignment algorithms [35–39] providing a candidate set of closed-loop eigenvalues and the associated eigenvectors that can achieve desired characteristics. For example, robust pole placement can indeed be realized by minimizing the condition number of the eigenvector matrix which renders the eigenvalues as insensitive to perturbations in the closed-loop system matrices as possible. Since each DF mode is associated with a purely imaginary pole, the sensitivity of DF modes to perturbations in the system parameters is minimized accordingly if the poles are robustly assigned. We follow this rule to solve the robust placement problem by assigning the poles to the imaginary axis for the synthesis of DF modes in the storage stage within the closed-loop quantum system. A coherent quantum controller is thus designed and included in the feedback loop, whose physical realizability conditions have been taken into account, with desired robustness achieved by assigning the imaginary poles appropriately. In addition, an analytical form of solution has been obtained for the design of such a coherent controller, which may involve optional input and output channels. The memory modes that are DF in the storage stage can be tuned to couple to the associated transportation channels during the writing and reading stages.

Notations. In this paper, $*$ is used to indicate the adjoint X^* of an operator X, as well as the complex conjugate $z^* = x - \mathrm{i}y$ of a complex number $z = x + \mathrm{i}y$ ($\mathrm{i} = \sqrt{-1}$ and x, y are real). $|z|$ denotes the modulus of z. The conjugate transpose $A^\dagger$ of a matrix $A = \{a_{ij}\}$ is defined by $A^\dagger = \{a_{ji}^*\}$. $\|A\|_p$ denotes the p-norm of a matrix A. The commutator of two operators X, Y is defined by $[X, Y] = XY - YX$. I_n ($n \in \mathbb{N}$) denotes the n-dimensional identity matrix.

2 Linear quantum passive systems

The dynamics of an open quantum system can be described by the triplet (S, L, H). The unitary matrix S is a scattering matrix, and the column vector L with operator entries is defined as the coupling operator. S and L together specify the interface between the system and its external environments. In addition, the self-adjoint operator H denotes the Hamiltonian (self-energy) of the system. In the Schrödinger picture of quantum mechanics, the master equation for a density state ρ of the quantum system can be formulated from the triplet as

$$d\rho = \left(\mathrm{i}[\rho, H] + \mathcal{L}^*(\rho)\right) dt, \tag{1}$$

where the scattering matrix S is assumed to be identity without loss of generality. By contrast, given an operator X defined on the Hilbert space H, the corresponding Heisenberg-picture evolution is given by

$$dX = \left(\mathcal{L}(X) - \mathrm{i}[X, H]\right) dt + dW^\dagger [X, L] + \left[L^\dagger, X\right] dW, \tag{2}$$

with

$$\mathcal{L}(X) = \frac{1}{2} L^\dagger [X, L] + \frac{1}{2}\left[L^\dagger, X\right] L. \tag{3}$$

Here $\mathcal{L}(\cdot)$ is called the Lindblad superoperator (Lindbladian), and $\mathcal{L}^*(\cdot)$ denotes the adjoint of the Lindbladian. The operator W is defined on a special Hilbert space F called Fock space. When the input fields are in the vacuum states, the fundamental annihilation process W and creation process $W^\dagger$ are quantum Wiener processes satisfying the quantum Itô rule

$$dW\, dW^\dagger = I_m\, dt, \tag{4}$$

if the number of inputs is m (namely $W = [W_1 \ \cdots \ W_m]^T$) [40–43].

In this paper we focus on linear quantum passive systems. A linear quantum passive system can be modeled by a set of harmonic oscillators coupled to bosonic fields, where interactions between the system and the field are passive. In other words, the dynamics of a linear quantum passive system are completely characterized by its annihilation operators since no external source of quanta is required to be implemented. Such a system can be realized using only passive optical components like phase shifters, beam splitters and mirrors, widely applied in linear optical quantum memories. To be specific, the type of passive systems defined in terms of only annihilation operators can be described by the following stochastic differential equations that can be derived from Eq. (2)

$$\begin{aligned} \dot{a}(t) &= Aa(t) - C^\dagger b(t), \qquad a(0) = a_0, \\ b_{\mathrm{out}}(t) &= Ca(t) + b(t). \end{aligned} \tag{5}$$

Here $a(t) = [a_1(t) \ \cdots \ a_n(t)]^T$ is a vector of annihilation operators, with the j-th mode represented by $a_j(t)$ satisfying the canonical commutation relations $[a_j, a_k^*] = 1$ for $j = k$ and $[a_j, a_k^*] = 0$ for $j \neq k$. The function of a_j is to annihilate one photon in the j-th mode, while the function of a_k^* is to create one photon in the k-th mode according to the underlying physics. Indeed, the coefficient matrices A and C in Eq. (5) can be obtained from Eq. (2)

by letting $X = a$ and

$$H = a^\dagger \Omega a, \qquad L = Ca,$$

respectively. Here $\Omega \in \mathbb{C}^{n\times n}$ is a Hermitian matrix ($\Omega^\dagger = \Omega$), $A \in \mathbb{C}^{n\times n}$, and $C \in \mathbb{C}^{m\times n}$ are complex matrices. Furthermore, we have

$$A = -\mathrm{i}\Omega - \frac{1}{2}C^\dagger C, \tag{6}$$

and thus the condition for physical realizability can be established as follows

$$A + A^\dagger + C^\dagger C = 0. \tag{7}$$

The term physical realizability corresponds to the algebraic physical realizability condition given by Eq. (7), which stems from the non-commuting nature of quantum observables, associated with stochastic differential equations Eq. (5) describing the dynamics of physically meaningful open quantum systems [24, 44]. Or rather, the physical realizability condition guarantees that the commutation relations can be preserved as required by unitary evolution in quantum mechanics. Unlike the classical case where every system model characterized by stochastic differential equations can be physically realized in principle, not every system model of the form like Eq. (5) can be physically realized as linear quantum systems corresponding to quantum harmonic oscillators unless the physical realizabity condition is satisfied; see e.g. [24, 43] and the references therein for more details. Moreover, $b(t) = [b_1(t) \ \cdots \ b_m(t)]^T$ is a vector of bosonic annihilation operators defined on F satisfying

$$W(t) = \int_0^t b(s)\,ds, \qquad \left[b_j(t), b_j^*(s)\right] = \delta(t-s),$$

where $\delta(\cdot)$ is the Dirac delta function. Similarly, $b_j(t)$ annihilates one photon in the j-th input field at time t.

When there exist unknown inputs to the linear quantum passive system, the following stochastic equations can be used to describe the altered dynamics:

$$\begin{aligned} \dot{a}(t) &= Aa(t) - C^\dagger \tilde{b}(t), \qquad a(0) = a_0, \\ b_{\mathrm{out}}(t) &= Ca(t) + \tilde{b}(t). \end{aligned} \tag{8}$$

When there are unknown signals in the bosonic fields, this type of uncertainty could be described by an adapted process β, representing signals defined on a space distinct from that of a and the quantum Wiener processes interacting the quantum plant via system-field interplay. This amounts to $\tilde{b} = b + \beta$. To be more precise, the signal β, which is also assumed to be bounded, commutes with $\tilde{b}$ and a for all $t \geq 0$ [21, 24].

3 Coherent robust pole placement in the presence of unknown inputs

Inspired by the work [21, 30, 31], in this section, we will introduce the technique dealing with coherent robust pole placement including unknown input signals. Consider a linear quantum passive system as the plant whose dynamics are governed by the following

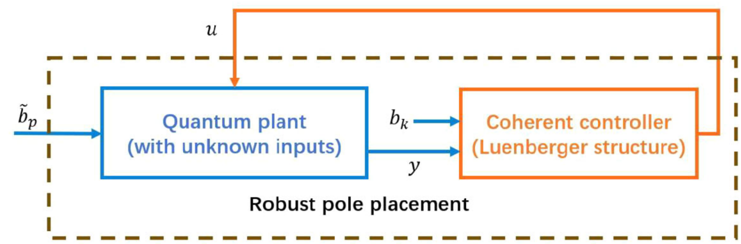

Figure 2 Realization of robust pole placement using a Luenberger structure coherent quantum controller for a linear quantum passive system as the quantum plant, in the presence of unknown inputs

equations

$$\dot{a}_p(t) = A_p a_p(t) - C_p^\dagger \tilde{b}_p(t) - C_f^\dagger u(t), \tag{9}$$

$$y(t) = C_p a_p(t) + \tilde{b}_p(t), \tag{10}$$

with the coefficient matrices denoted by $A_p \in \mathbb{C}^{n\times n}$, $C_p \in \mathbb{C}^{m_p\times n}$ and $C_f \in \mathbb{C}^{m_k\times n}$. Here $\tilde{b}_p(t) = b_p(t) + \beta_p(t)$. In particular, the system matrix of the quantum plant can be determined by

$$A_p = -\mathrm{i}\Omega_p - \frac{1}{2}\left(C_p^\dagger C_p + C_f^\dagger C_f\right). \tag{11}$$

Equivalently, the physical realizability condition can be given by

$$A_p + A_p^\dagger + C_p^\dagger C_p + C_f^\dagger C_f = 0. \tag{12}$$

The structure of a quantum coherent controller is described by the following stochastic differential equations

$$\dot{a}_k(t) = \left(A_p - LC_p - C_f^\dagger K\right)a_k(t) - K^\dagger b_k(t) + Ly(t) - C_o^\dagger v(t), \tag{13}$$

$$u(t) = Ka_k(t) + b_k(t), \tag{14}$$

$$z(t) = -L^\dagger a_k(t) + y(t). \tag{15}$$

The coefficient matrix $K \in \mathbb{C}^{m_k\times n}$ is associated with the input-output channel described by Eq. (14), with the corresponding output $u(t)$ fed back to the plant, as shown in Fig. 2. The coefficient matrix $L \in \mathbb{C}^{n\times m_p}$ is associated with the input-output channel described by Eq. (15), which receives the input $y(t)$ from the plant. The coefficient matrix $C_o \in \mathbb{C}^{m_o\times n}$ is linked to an optional input $v(t)$, which may be involved to ensure the physical realizability of the controller. By interconnecting the plant and controller jointly via $u(t)$ and $y(t)$, the dynamical equations for the closed-loop system can be obtained as

$$\begin{bmatrix}\dot{a}_p(t)\\ \dot{e}(t)\end{bmatrix} = \begin{bmatrix}A_p - C_f^\dagger K & C_f^\dagger K\\ 0 & A_p - LC_p\end{bmatrix}\begin{bmatrix}a_p(t)\\ e(t)\end{bmatrix} + \begin{bmatrix}-C_p^\dagger & -C_p^\dagger & -C_f^\dagger & 0\\ -(C_p^\dagger + L) & -(C_p^\dagger + L) & -(C_p^\dagger + L) & C_o^\dagger\end{bmatrix}\begin{bmatrix}b_p(t)\\ \beta_p(t)\\ b_k(t)\\ v(t)\end{bmatrix}, \tag{16}$$

where the discrepancy $e(t)$ is defined as $e(t) = a_p(t) - a_k(t)$. Therefore it can be concluded that the poles of the closed-loop system are indeed the eigenvalues of $A_p - C_f^\dagger K$ and $A_p - LC_p$. The sufficient and necessary condition for the physical realizability of the coherent controller is then given by

$$A_p - LC_p - C_f^\dagger K = -\mathrm{i}\Omega_k - \frac{1}{2}K^\dagger K - \frac{1}{2}LL^\dagger - \frac{1}{2}C_o^\dagger C_o, \tag{17}$$

with $\Omega_k \in \mathbb{C}^{n\times n}$ being a Hermitian matrix.

In the rest of this paper, we assume that the pair $(A_p, C_f^\dagger)$ is controllable and the pair (A_p, C_p) is observable. According to the separation principle as discussed in [30], the eigenvalues of $A_p - C_f^\dagger K$ and $A_p - LC_p$ can be designed independently of each other given that $(A_p, C_f^\dagger)$ is controllable and (A_p, C_p) is observable.

Assumption 1 For linear quantum passive systems described by Eqs. (9) and (10) with unknown input signals, we assume that

- (A_p, C_p) is observable.
- $(A_p, C_f^\dagger)$ is controllable.

Bearing in mind Assumption 1, the precise definition of the coherent robust pole placement problem considered in this paper can be speculated as follows.

Definition 1 The robust pole placement problem is to find L and K such that the following conditions hold.

- $A_p - LC_p = X_o\Lambda_o X_o^{-1}$ with the eigenvector matrix $X_o \in \mathbb{C}^{n\times n}$ such that $\|X_o\|_2\|X_o^{-1}\|_2 = \chi_o$ for a condition number χ_o, where $\Lambda_o = \mathrm{diag}\{\lambda_{o_1},\ldots,\lambda_{o_n}\}$ with λ_{o_j} $(j \in [1,n]\cap\mathbb{N})$ being the desired poles.
- $A_p - C_f^\dagger K = X_k\Lambda_k X_k^{-1}$ with the eigenvector matrix $X_k \in \mathbb{C}^{n\times n}$ such that $\|X_k\|_2\|X_k^{-1}\|_2 = \chi_k$ for a condition number χ_k, where $\Lambda_k = \mathrm{diag}\{\lambda_{k_1},\ldots,\lambda_{k_n}\}$ with λ_{k_j} $(j \in [1,n]\cap\mathbb{N})$ being the desired poles.

In particular, when $\chi_j = 1$ $(j \in \{o,k\})$, X_j is said to be perfectly conditioned with maximal robustness.

Please note that according to Definition 1, the matrices $A_p - LC_p$ and $A_p - C_f^\dagger K$ are designed to be non-defective. In the following theorem, we show a concrete procedure to design a coherent controller to achieve the goal of robust pole placement for the quantum plant described by Eqs. (9) and (10) where unknown inputs exist.

Theorem 1 *A solution to the robust pole placement problem for the linear quantum passive system described by Eqs.* (9)-(10) *in the presence of unknown inputs, by means of a coherent controller described by Eqs.* (13)-(15), *is given by*

$$L = -C_p^\dagger, \tag{18}$$

$$K = T_K\hat{C} + C_f, \tag{19}$$

$$C_o = T_o\hat{C}, \tag{20}$$

if the following conditions hold

$$\hat{C}^\dagger \hat{C} = C_f^\dagger C_f - C_p^\dagger C_p \geq 0,$$
$$\Upsilon_K = -\mathrm{i}\Omega_p - \frac{1}{2}C_p^\dagger C_p - \frac{3}{2}C_f^\dagger C_f - C_f^\dagger T_K \hat{C} = X_k \Lambda_k X_k^{-1}.$$

Here $\hat{C} \in \mathbb{C}^{n\times n}$ *and* $T_K \in \mathbb{C}^{m_k \times n}$, *together with* $T_o \in \mathbb{C}^{m_o\times n}$, *satisfying*

$$T_K^\dagger T_K + T_o^\dagger T_o = 2I_n.$$

The best choice of T_K *is obtained when* $\chi_k = \|X_k\|_2 \|X_k^{-1}\|_2 = 1$. *The optional input-output channel is not needed when* $T_o = 0$.

Proof In the presence of unknown inputs, it is straightforward to require $L = -C_p^\dagger$. The physical realizability condition for the coherent controller can then be written as

$$\left(A_p + C_p^\dagger C_p - C_f^\dagger K\right) + \left(A_p + C_p^\dagger C_p - C_f^\dagger K\right)^\dagger + K^\dagger K + LL^\dagger + C_o^\dagger C_o = 0. \tag{21}$$

By substituting the physical realizability condition for the plant, i.e.,

$$A_p + A_p^\dagger = -\left(C_p^\dagger C_p + C_f^\dagger C_f\right)$$

into Eq. (21), one can obtain that

$$(K - C_f)^\dagger (K - C_f) + C_o^\dagger C_o = 2\hat{C}^\dagger \hat{C}, \tag{22}$$

where $\hat{C} \in \mathbb{C}^{n\times n}$ can be calculated by $\hat{C}^\dagger \hat{C} = C_f^\dagger C_f - C_p^\dagger C_p$ using Cholesky decomposition when $C_f^\dagger C_f - C_p^\dagger C_p \geq 0$. It can be verified that Eqs. (19) and (20) constitute a solution to Eq. (22). By appropriately choosing T_K, $\chi_k = |X_k\|_2 \|X_k^{-1}\|_2$ can be made close to 1 or exactly 1.

Especially, if $T_K^\dagger T_K = 2I_n$, the optional input $v(t)$ does not have to be fed to the coherent controller. □

Please note that the inner model (as shown in Fig. 1) is not considered yet in the system setup in this section.

Remark 1 In particular, when $m_p \geq n$, one can make

$$T_K^\dagger T_K = |s_K|^2 I_n,$$

with s_K a complex number. Then C_o can be determined by $C_o = T_o \hat{C}$ with

$$T_o^\dagger T_o = \left(2 - |s_K|^2\right) I_n.$$

4 Synthesis of robust DF modes in the presence of unknown inputs

As aforementioned, either $(A_p, C_f^\dagger)$ being controllable or (A_p, C_p) being observable implies that there exist no DF modes in the plant. Since the synthesis of robust DF modes is equivalent to placing the target poles of linear quantum passive systems at the imaginary axis [29], a natural application of Theorem 1 regarding the synthesis of DF modes robustly in the presence of unknown inputs is presented in the following theorem.

Theorem 2 *DF modes can be synthesized robustly for the linear quantum passive system described by Eqs.* (9)-(10) *in the presence of unknown inputs, by means of a coherent controller described by Eqs.* (13)-(15), *if*

- *at least one assigned pole* λ_{j^*} $(j^* \in \{k_1, \dots, k_n\})$ *is purely imaginary with the condition number* χ_k;
- *the other poles* λ_j $(\{j \neq j^* | j \in \{k_1, \dots, k_n\}\})$ *which are not purely imaginary have negative real parts.*

Proof According to Theorem 1, since $L = -C_p^\dagger$, one can have that

$$A_p - LC_p = A_p + C_p^\dagger C_p = -\mathrm{i}\Omega_p + \frac{1}{2}\left(C_p^\dagger C_p - C_f^\dagger C_f\right). \tag{23}$$

In addition, it is required that $C_f^\dagger C_f - C_p^\dagger C_p \geq 0$ in Theorem 1. As $A_p - LC_p$ is assumed to be non-defective and the Hermitian matrix Ω_p is usually defined to be diagonal, the eigenvalues of $A_p - LC_p$ are then guaranteed to be imaginary or have negative real parts.

In terms of K, one can make the diagonal elements in Λ_k have negative real parts except for at least one being purely imaginary. Then K can be determined by following the procedure given in Theorem 1. □

In addition, the controller matrix K can be tuned in different stages. Specifically, in the holding stage, K can be chosen to make $A_p - C_f^\dagger K$ have at least one imaginary eigenvalue with the others having negative real parts. In the writing or reading stage, $A_p - C_f^\dagger K$ is made to only have eigenvalues in the left half of the complex plane. In addition, the real parts of these poles can be made larger in order to make the dynamics faster.

By contrast to the results in [30], our main goal here is to synthesize DF modes robustly in the presence of unknown inputs. Therefore, it is of much importance to make the condition number χ_k exactly 1. The following corollary shows that further restriction on the choice of K will be introduced if perfect robustness can be achieved.

Corollary 1 *The condition number* $\chi_{k(o)}$ *equals* 1 *if and only if* $X_{k(o)}X_{k(o)}^\dagger = X_{k(o)}^\dagger X_{k(o)} = \sigma^2 I_n$ *where* $\sigma > 0$ *is a positive real number.*

Proof According to Definition 1, when the condition number equals 1, perfect robustness of DF modes can be obtained by making at least one diagonal element of $\Lambda_{k(o)}$ purely imaginary with the others all having negative real parts. Namely,

$$\chi_{k(o)} = \|X_{k(o)}\|_2 \left\|X_{k(o)}^{-1}\right\|_2 = 1, \tag{24}$$

or equivalently,

$$\chi_k(o) = \frac{\sigma_{\max}}{\sigma_{\min}} = 1, \tag{25}$$

where $\sigma_{\max}$ and $\sigma_{\min}$ are the largest and smallest singular values of the eigenvector matrix $X_{k(o)}$ respectively. Therefore, by setting $\sigma_{\max} = \sigma_{\min} = \sigma$, the eigenvector matrix $X_{k(o)}$ can be written as

$$X_{k(o)} = \sigma UV^{\dagger}, \tag{26}$$

where both U and V are unitary matrices, with σ a positive real number denoting the identical singular value. It can then be concluded that the eigenvector matrix equals 1 if and only if

$$X_{k(o)}X_{k(o)}^{\dagger} = X_{k(o)}^{\dagger}X_{k(o)} = \sigma^2 I_n. \tag{27}$$

□

Remark 2 According to Theorem 2, it can be obviously seen that the presence of unknown inputs brings in new challenge, which mathematically puts more constraints on the choice of L. Consequently, There is not adequate degree of freedom for assigning the poles of $A_p - LC_p = -\mathrm{i}\Omega_p + \frac{1}{2}(C_p^{\dagger}C_p - C_f^{\dagger}C_f)$ at desired locations.

In order to allow for more degrees of freedom for synthesizing robust memory modes, in the following section an auxiliary mode will be included to generate an augmented system model.

5 Inner model method to weaken the constraints for synthesizing robust memory modes

In order to weaken coherent controller design constraints when unknown inputs are taken into account, in this section we employ the method of inner model as discussed in [21] to allow for more degrees of freedom regarding the choices of controller coefficient matrices. The resulting augmented system, as shown in Fig. 1, enables us to be less dependent on the structure of the original quantum plant. In more concrete terms, we introduce a new variable defined by the integral of β_p over time as

$$\dot{u}_a(t) = \beta_p. \tag{28}$$

We then consider the following augmented system including the auxiliary signal $u_a(t)$ as one of the state variables

$$\begin{bmatrix} \dot{a}_p(t) \\ \dot{u}_a(t) \end{bmatrix} = \begin{bmatrix} A_p & Q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_p(t) \\ u_a(t) \end{bmatrix} - \begin{bmatrix} C_p^{\dagger} \\ -I_{m_p} \end{bmatrix} \beta_p - \begin{bmatrix} C_p^{\dagger} \\ 0 \end{bmatrix} b_p - \begin{bmatrix} C_f^{\dagger} \\ 0 \end{bmatrix} u, \tag{29}$$

$$y = \begin{bmatrix} C_p & N \end{bmatrix} \begin{bmatrix} a_p(t) \\ u_a(t) \end{bmatrix} + b_p + \beta_p, \tag{30}$$

where the coefficient matrices Q and N create new degrees of freedom for one to design DF modes. To simplify the derivation, we use the following notations

$$\dot{\bar{a}}_p(t) = \bar{A}\bar{a}_p(t) - \bar{B}_1\beta_p - \bar{B}_2 b_p - \bar{B}_3 u, \tag{31}$$

$$y = \bar{C}\bar{a}_p(t) + b_p + \beta_p, \tag{32}$$

where the coefficient matrices $\bar{A} = \begin{bmatrix} A_p & Q \\ 0 & 0 \end{bmatrix}$, $\bar{B}_1 = \begin{bmatrix} C_p^\dagger \\ -I_{m_p} \end{bmatrix}$, $\bar{B}_2 = \begin{bmatrix} C_p^\dagger \\ 0 \end{bmatrix}$, $\bar{B}_3 = \begin{bmatrix} C_f^\dagger \\ 0 \end{bmatrix}$, $\bar{C} = \begin{bmatrix} C_p & N \end{bmatrix}$. The following theorem shows that DF modes can be synthesized in the augmented plant-controller feedback system even if a DF mode cannot be synthesized in the original closed-loop system. Moreover, the unknown inputs can be tracked via a coherent observer-based controller in such a scenario, and thus further robustness can be provided in this sense.

Theorem 3 *In the presence of unknown inputs, even if a DF mode cannot be synthesized robustly for the linear quantum passive system described by Eqs.* (9)-(10) *in the closed-loop, by incorporating the inner model, a Luenberger-type coherent controller can be designed for the resulting augmented system described by Eqs.* (29)-(30) *in order to synthesize robust memory modes.*

Proof The dynamics for the Luenberger-type coherent controller can be written as

$$\dot{\bar{a}}_k(t) = \big(\bar{A} - \bar{L}\bar{B}_1^\dagger - \bar{B}_3\bar{F}\big)\bar{a}_k(t) - \bar{F}^\dagger \bar{b}_k + \bar{L}y - \bar{C}_o^\dagger v. \tag{33}$$

Similar to our analysis in the previous section, we have that

$$\dot{\bar{a}}_p(t) = (\bar{A} - \bar{B}_3\bar{K})\bar{a}_p(t) + \bar{B}_3\bar{F}\bar{e}(t) - \bar{B}_1\beta_p - \bar{B}_2 b_p - \bar{B}_3 b_k(t), \tag{34}$$

and

$$\begin{aligned}\dot{\bar{a}}_p(t) - \dot{\bar{a}}_k(t) = {} & (\bar{A} - \bar{L}\bar{C})\big(\bar{a}_p(t) - \bar{a}_k(t)\big) + \big(\bar{F}^\dagger - \bar{B}_3\big)b_k(t) \\ & - (\bar{B}_1 + L)\beta_p - (\bar{B}_2 + L)b_p + \bar{C}_o^\dagger v(t).\end{aligned}$$

In order to eliminate the influence of unknown inputs in the system, one can choose $\bar{B}_1 + L = 0$. The physical realizability condition for the coherent controller can be equivalently written as

$$\big(\bar{A} + \bar{B}_1\bar{B}_1^\dagger - \bar{B}_3\bar{F}\big) + \big(\bar{A} + \bar{B}_1\bar{B}_1^\dagger - \bar{B}_3\bar{F}\big)^\dagger + \bar{F}^\dagger\bar{F} + \bar{L}\bar{L}^\dagger + \bar{C}_o^\dagger\bar{C}_o = 0. \tag{35}$$

In this case, one can further compute that

$$\bar{A} + \bar{B}_1\bar{C} = \begin{bmatrix} A_p & Q \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} C_p^\dagger \\ -I_{m_p} \end{bmatrix}\begin{bmatrix} C_p & N \end{bmatrix} = \begin{bmatrix} A_p + C_p^\dagger C_p & Q + C_p^\dagger N \\ -C_p & -N \end{bmatrix}. \tag{36}$$

Apparently, Eq. (36) offers us more freedom to assign the poles to desired locations. Especially, one can choose $Q + C_p^\dagger N = 0$, and thus the augmented system matrix $\bar{A} + \bar{B}_1\bar{C}$ turns into

$$\begin{bmatrix} A_p + C_p^\dagger C_p & 0 \\ -C_p & -N \end{bmatrix}.$$

We can then locate the eigenvalues of N either at the imaginary axis or in the left half of the complex plane. Therefore, N can not only increase the freedom of degrees, but N can also help us assign the poles of the augmented system on the imaginary axis.

On the other hand, please note that

$$\bar{A} - \bar{B}_3\bar{K} = \begin{bmatrix} A_p & Q \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} C_f^\dagger \\ 0 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix} = \begin{bmatrix} A_p - C_f^\dagger K_1 & Q - C_f^\dagger K_2 \\ 0 & 0 \end{bmatrix}.$$

We can then choose K_2 to make $Q - C_f^\dagger K_2 = 0$ such that a_p is decoupled from u_a. And since $(A_p, C_f^\dagger)$ is controllable, K_1 can be appropriately chosen by assigning the eigenvalues of $A_p - C_f^\dagger K_1$ either at the imaginary axis or in the left half of the complex plane. □

It is worth mentioning that N can also be chosen to be Hurwitz. The controller matrix K_1 can be tuned in different stages. Specifically, in the holding stage, K_1 can be chosen to make $A_p - C_f^\dagger K_1$ have at least one imaginary eigenvalue with the others having negative real parts. The condition number can be made 1 following the rules in Corollary 1. In the writing or reading stage, $A_p - C_f^\dagger K_1$ is made to only have eigenvalues in the left half of the complex plane. In addition, the real parts of these poles can be made larger in order to make the dynamics faster.

6 Examples of synthesizing robust memory modes for optical cavities

In this section we consider optical systems whose structure can be illustrated by Fig. 3, in which the plant cavity has in general no DF modes due to its coupling to the external environment via the optical field signal $\tilde{b}_p$. A controller cavity, which additionally has input b_k, is then designed to generate robust DF modes that are shared in the composite plant-controller coherent feedback control system.

Specifically, the plant in the following example is a linear quantum passive system whose dynamics correspond to a single-mode cavity.

Example 1 The positive real numbers Ω and κ are used to quantify the cavity Hamiltonian and coupling strength between the cavity and optical fields. The coefficient matrices describing the quantum plant characterized by Eqs. (9) and (10) are given by

$$A_p = -(\mathrm{i}\Omega_p + 2\kappa), \qquad C_p = \sqrt{\kappa}, \qquad C_f = \sqrt{3\kappa}. \tag{37}$$

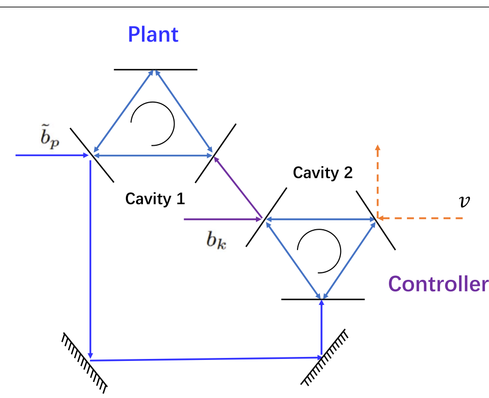

Figure 3 Coherent feedback network for the synthesis of robust memory modes concerning an optical cavity as the quantum plant, where the input $\tilde{b}_p$ to the plant contains uncertainty. The coherent quantum controller is another optical cavity interconnecting with the original plant via optical fields. Optional input v may be fed to the coherent controller to satisfy the physical realizability condition if necessary

It is not difficult to verify that (A_p, C_p) is observable and $(A_p, C_f^\dagger)$ is controllable. Therefore, there does not exist a DF mode in the original quantum plant with coefficient matrices provided in Eq. (37).

Following the procedure in Theorems 1 and 2, we are supposed to choose $L = -\sqrt{\kappa}$, and thus in this case

$$A_p - LC_p = A_p + C_p^\dagger C_p = -\mathrm{i}\Omega_p - \kappa.$$

In order to create a DF mode in the feedback control system via a coherent controller, one has to choose $\Re(K) = -\sqrt{\frac{4\kappa}{3}}$ such that $A_p - C_f^\dagger K$ can be made purely imaginary. However, according to Theorems 1 and 2, we have that

$$2\hat{C}^\dagger\hat{C} - (K - C_f)^\dagger(K - C_f) \le -4.33\kappa < 0,$$

which indicates the controller is not physically realizable. Therefore, DF modes cannot be synthesized concerning the original quantum plant in Example 1 in the presence of unknown input, unless the augmented system is considered by allowing for more degrees of freedom.

One now can follow the procedure in Theorem 3 to synthesize robust DF modes by incorporating an inner model. For example, we can choose $N = \mathrm{i}\Omega_N$ and $Q = -C_p^\dagger N$, it can then be found that

$$\bar{A} + \bar{B}_1\bar{C} = \begin{bmatrix} -i\Omega_1 - \kappa & Q + C_p^\dagger N \\ -\sqrt{\kappa} & -N \end{bmatrix} = \begin{bmatrix} -i\Omega_1 - \kappa & 0 \\ -\sqrt{\kappa} & -i\Omega_N \end{bmatrix}.$$

In addition, regarding the choice of the controller parameter $K = [\,\kappa_1\ \kappa_2\,]$, we can simply choose $K_2 = C_f^{-1}Q$ in this example by referring to Theorem 3. K_1 can then be appropriately chosen by assigning $A_p - C_f^\dagger K_1$ either at the imaginary axis or in the left half of the complex plane, provided that the physical realizability conditions are satisfied. Hence, by introducing the inner model method, we are enabled to synthesize DF modes for the storage stage in the coherent feedback control system. On the other hand, N and $A_p - C_f^\dagger K_1$ in this example can be easily tuned to have negative real parts for the writing/reading stage.

In Example 2, we aim to show tuning of controller parameters which leads to different values of the condition number, towards robustly synthesizing DF modes in the presence of unknown inputs.

Example 2 Consider a linear quantum passive system as the quantum plant physically realized as a two-mode cavity. Without loss of generality, let

$$\Omega = \begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{bmatrix}, \tag{38}$$

and $\Omega_1 > \Omega_2$. Here Ω_1, Ω_2 represent the optical frequencies of the two cavity modes. We assume that the coupling constant κ is the same for the input and output fields. The parameters Ω_1, Ω_2 and κ are positive real numbers. The coefficient matrices describing the quantum plant can thus be written as

$$A_p = \begin{bmatrix} -(i\Omega_1 + \kappa) & -\kappa \\ -\kappa & -(i\Omega_2 + \kappa) \end{bmatrix}, \qquad C_p = \begin{bmatrix} \sqrt{\kappa} & \sqrt{\kappa} \end{bmatrix}, \qquad C_f = \begin{bmatrix} \sqrt{\kappa} & \sqrt{\kappa} \end{bmatrix}. \tag{39}$$

It can be easily verified that

$$\det \begin{bmatrix} C_p \\ C_p A_p \end{bmatrix} = i\kappa(\Omega_1 - \Omega_2) \neq 0,$$

which indicates that (A_p, C_p) is observable. Also, $(A_p, C_f^\dagger)$ is controllable since

$$\det\begin{bmatrix} C_f^\dagger & A_p C_f^\dagger \end{bmatrix} = i\kappa(\Omega_1 - \Omega_2) \neq 0.$$

As a result, there does not exist a DF mode in the original plant. For the synthesis of robust DF modes in the presence of unknown inputs with the quantum coherent controller as stated in Theorem 3, we parameterize L and K_1 as

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}, \qquad K_1 = \begin{bmatrix} k_{11} \\ k_{12} \end{bmatrix}^T,$$

with l_1, l_2, k_{11} and k_{12} complex numbers. Then we have

$$A_p - LC_p = \begin{bmatrix} -(i\Omega_1 + \kappa) - l_1\sqrt{\kappa} & -\kappa - l_1\sqrt{\kappa} \\ -\kappa - l_2\sqrt{\kappa} & -(i\Omega_2 + \kappa) - l_2\sqrt{\kappa} \end{bmatrix},$$
$$A_p - C_f^\dagger K_1 = \begin{bmatrix} -(i\Omega_1 + \kappa) - k_1\sqrt{\kappa} & -\kappa - k_2\sqrt{\kappa} \\ -\kappa - k_1\sqrt{\kappa} & -(i\Omega_2 + \kappa) - k_2\sqrt{\kappa} \end{bmatrix}.$$

Following the procedure in Theorem 1 and Corollary 1, an obvious choice of K_1 to achieve the minimum condition number of $\chi_k = 1$ is $K_1 = [\,-\sqrt{\kappa}\ -\sqrt{\kappa}\,]$ with $T_K = [\,-1\ -1\,]$. In this case, we obtain the perfectly-conditioned imaginary poles as

$$\Lambda_k = \begin{bmatrix} -i\Omega_1 & 0 \\ 0 & -i\Omega_2 \end{bmatrix}. \tag{40}$$

In order to eliminate the influence of unknown inputs, the matrix L is chosen to be $L = [\,-\sqrt{\kappa}\ -\sqrt{\kappa}\,]^T$. The matrices Q, N and K_2 can be appropriately determined by following the rules given in Theorem 3, so that the poles of the closed-loop feedback systems are located either at the imaginary axis or in the left half of the complex plane. Therefore, by choosing $K_1 = [\,-\sqrt{\kappa}\ -\sqrt{\kappa}\,]$, the imaginary poles $-i\Omega_1$ and $-i\Omega_2$ corresponding to robust DF modes are perfectly conditioned.

In order to see the influence of K_1 on the condition number χ_k, we let

$$k_{11} = -\sqrt{\kappa} + \frac{\epsilon(\Omega_1 - \Omega_2)}{\sqrt{\kappa}}, \qquad k_{12} = -\sqrt{\kappa},$$

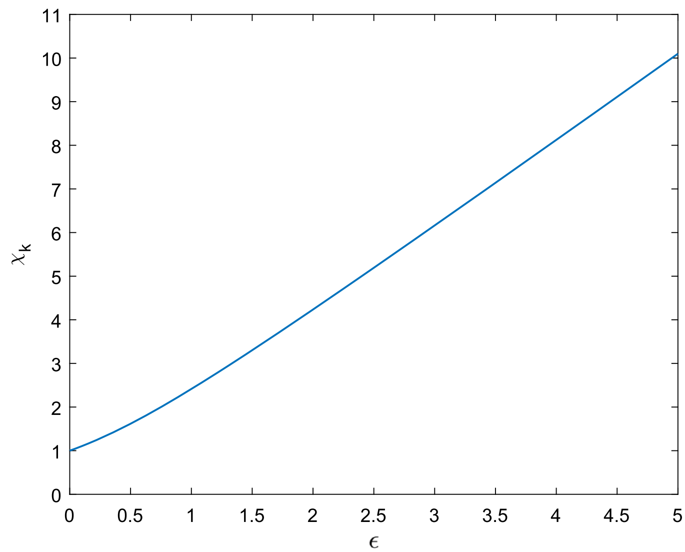

Figure 4 Changes of the condition number χ_k when ϵ (a controller parameter) varies according to Eq. (41). As ϵ increases, the robustness characterized by χ_k becomes weaker

with ϵ being a non-negative real number that can be varied. Then we have that

$$A_p - C_f^{\dagger} K_1 = \begin{bmatrix} -\epsilon(\Omega_1 - \Omega_2) - \mathrm{i}\Omega_1 & 0 \\ -\epsilon(\Omega_1 - \Omega_2) & -\mathrm{i}\Omega_2 \end{bmatrix},$$

and the corresponding condition number

$$\chi_k = \epsilon + \sqrt{\epsilon^2 + 1}. \tag{41}$$

It can be obviously seen from Fig. 4 that as ϵ increases, the condition number χ_k moves further from 1. The strongest robustness is achieved with respect to the imaginary pole $-\mathrm{i}\Omega_2$ when $\epsilon = 0$, which is indeed indicated by Corollary 1.

7 Conclusions

In this paper, we have proposed a coherent feedback control scheme with an explicit design method for the synthesis of robust DF modes concerning linear quantum passive systems in the presence of unknown inputs. To be more specific, if the original system (referred to as the quantum plant) does not possess any DF modes, a coherent quantum controller can be constructed whose output is fed to the plant, with the aim of assigning the poles of the closed-loop system onto the imaginary axis or into the left half of the complex plant robustly. Please note that the imaginary poles directly correspond to DF modes. In particular, robust DF modes, quantified by the condition number, mean that the purely imaginary poles are robust against perturbations to the system parameters by appropriately designing the controller, which can thus minimize the information leakage of DF modes to the environment in practice. In Example 1, by following the rules in Theorems 1-3, it has been demonstrated that even if a DF mode cannot be synthesized by the conventional coherent feedback scheme, DF modes can be synthesized in the presence of unknown inputs by employing the inner model method. It has been shown in Example 2 that it is convenient to improve the robustness of DF modes by adjusting the controller parameters which tune the condition number corresponding to the imaginary poles, from an ill-conditioned case to a well-conditioned (perfect) case. Furthermore, by appropriately tuning the controller parameters, the robust DF modes can be switched on and off, which allows for storing as well as writing and reading of quantum information. It is thus promising that the approach proposed in this paper for the synthesis of robust memory modes can be applied to facilitate secure storage and communication of quantum information.

Author contributions
Conceptualization, Z. M. and Y. P.; methodology, Z. M., Y. P. and X. C.; formal analysis, Z. M., Y. P. and X. C.; data curation, Z. M., X. C., Y. P. and Q. G.; writing-original draft preparation, Z. M. and Y. P.; writing-review and editing, Z. M., Y. P. and Q. G.; visualization, Z. M., Y. P. and X. C.; supervision, Z. M.; funding acquisition, Z. M., Y. P. and Q. G..

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Competing interests
The authors declare no competing interests.

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