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Mathematical sense making of quantum phenomena using Dirac notation: its effect on secondary school students' functional thinking about photons

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Abstract

Previous research has consistently demonstrated that students often possess an inadequate understanding of fundamental quantum optics concepts, even after formal instruction. Findings from physics education research suggest that introducing a mathematical formalism to describe quantum optical phenomena may enhance students' conceptual understanding of quantum optics. This paper investigates whether using formal descriptions of quantum optics phenomena – such as photon anticorrelation at a beamsplitter or single-photon interference in a Michelson interferometer – expressed in Dirac notation, can support secondary school students in developing functional thinking about photons. To investigate this, we conducted a clusterrandomized field study, comparing the improvement in functional thinking between 67 students in the intervention group, who were taught using both qualitative and quantitative reasoning, and 66 students in the control group, who were taught using only qualitative reasoning. The results indicate that mathematical formalism can indeed promote functional thinking about photons. However, the comparison between the intervention and control groups revealed that the control group exhibited a greater increase in functional thinking than the intervention group. In response to these findings, we conducted a follow-up study aimed at gaining a deeper understanding of the cognitive load associated with both approaches. Specifically, we compared the intrinsic and extraneous cognitive load of 71 students in the intervention group with those of 65 students in the control group. The data analysis revealed that the two groups had statistically significant differences in intrinsic cognitive load while the extraneous cognitive load did not difer statistically significant, indicating a higher mental effort associated to the quantitative reasoning.

Keywords: Quantum physics; Photon; Dirac notation; Secondary school

1 Introduction

Quantum physics (QP) is currently incorporated into the educational curricula of numerous countries [1] and it often falls upon physics teachers to plan the way they approach QP in upper secondary school education. To do so, they can rely on research-based teaching concepts from quantum eduation research [2–12]. Additionally, quantum education research has provided a comprehensive overview of misconceptions that students encounter when being confronted with quantum topics: For instance, students have been found to think about electrons as tiny spherical entities [13] or understand electron spin as a rotation of a tiny spherical entities [14], or photons as light particles moving along sinusodial-trajectories [15]. Most of these learning difficulties can be traced back to students' having not developed a sufficiently sophisticated understanding of the underlying (quantum) models [16].

Despite great teaching approaches, learners frequently develop misconceptions about quantum objects because a single quantum object cannot be described and also visualised by a continuous temporal and spatial way.

To gain a deeper understanding of these misconceptions, previous research has developed a range of theoretical frameworks to describe cognitive processes and mental models [17, 18] (for further insight, see Sect. 2.1). Ubben and Heusler investigated students' conceptions about the atomic shell, and identified a two-dimensional structure of the mental models held by students and designated these two dimensions as "Fidelity of Gestalt" and "Functional Fidelity" [19]. Subsequent studies have evidenced that this two-dimensional structure can be used to describe learners' mental models of the photon [17], and that there is a correlation between the conceptual understanding of QP and the degree of Functional Fidelity in students' thinking about photons [20]. In consideration of these observations, it can be deduced that the enhancing of the Functional Fidelity in the thinking of the students should be regarded as a principal aim of QP instruction. However, physics education research has to investigate the implementation of teaching practices that effectively promote the degree of Functional Fidelity in students thinking about the photon. According to Ubben et al. [12, 20] promoting the degree of functional fidelity in the learner's thinking is accompanied by various abstraction processes. Since mathematics intrinsically aims at describing scientific concepts in an abstract manner it seems suitable that an introduction of a formalistic description rather than only using qualitative arguments may be a beneficial for developing mental models that exhibit higher degrees of functionality. This topic has been largely unstudied in the field of physics education research (see Sect. 2.2), as the fundamental principles of mathematical formalisms used to describe quantum phenomena extend beyond the scope of traditional school mathematics.

This conflict constitutes the starting point of our research as we investigate whether and to what extent the introduction of a mathematical formalism facilitates the transition from a mental model with a predominant degree of gestalt thinking to a mental model with a predominant degree of Functional Fidelity. To this end, Hennig et al. [2] developed a learning sequence with a formalistic approach to QP at its core in prior research [2]. It extends the qualitative treatment of single-photon experiments, such as anticoincident measurement results at a beam splitter or single-photon interference in a Michelson interferometer, with a mathematical description using Dirac notation. This approach does not require concepts of vector calculus or complex numbers [2]. Subsequently, the learning sequence was piloted using an acceptance survey with $n = 14$ learners, indicating that the majority of the instructional elements were well accepted [21]. In (quantum-)physics instruction, learners should be supported in abandoning such misconceptions. As already described, the introduction of a mathematical formalism might increase the degree of Functional Fidelity in the learner's mental model of the photon (see Sect. 2.1 and Sect. 2.2) and may be fruitfull to overcome such misconceptions. To investigate this research desideratum, we are conducting a field study utilizing the piloted learning sequence. Our paper contributes to addressing the question of whether learners develop a mental model with a higher degree of Functional Fidelity with the help of mathematical formalism. Therefore we present the results of a clusterrandomized field study with seconary school students, in which an experimental group was introduced to a mathematical formalism in addition to the qualitative instructions in a QP class, while the control group received only a qualitative reasoning.

2 Research background

2.1 Mental models

An individual's mental model can be understood as a cognitive representation of an object that is used to understand the functionality of complex systems [22]. It can therefore be seen as an individual interpretation that is assigned to a phenomenon or concept [23]. Once an individual is unable to explain a phenomenon using an existing mental model, the model is modified to accommodate the new information [24]. A variety of theoretical frameworks have been developed with the purpose of describing mental models and the manner in which they are constructed (e.g. [18, 19]). One of these frameworks has been developed within the context of QP [19]. In their research, Ubben and Heusler identified a two-dimensional structure that underlies learners' mental models of the atomic shell [19]. The authors called one dimension the *Fidelity of Gestalt*, which describes "how far the mental models [...] were understood as exact visual representations of phenomena of exact depictions of how things look" [19, p. 1356]. The other dimension is regarded as *Functional Fidelity* and indicates "how far the mental models [...] were thought of as appropriate descriptions of how phenomena work – what abstract concepts underly the corresponding models" [19, p.1356]. While these dimensions were originally discovered in the context of the atomic hull, previous research further revealed that this two-factor model is also applicable in the context of learners' mental models of the quantum object photon [17]. In a subsequent study, the authors found a positive correlation between the degree of Functional Fidelity in learners' mental models and their conceptual understanding of quantum optical concepts [20]. This positive correlation was then leveraged to propose the existence of a hierarchy, whereby four distinct types of mental models can be identified, representing varying degrees of Functional Fidelity or Fidelity of Gestalt [19]:

1. If both the Fidelity of Gestalt and the Functional Fidelity of a mental model are deemed to be low, it is regarded as an undeveloped type.
2. If the degree of Functional Fidelity of a mental model is low and the degree of Fidelity of Gestalt is high, the mental model is perceived as an exact representation of the underlying reality. This type of mental model is called architectural.
3. If both the Functional Fidelity and the Fidelity of Gestalt of a mental model are high, it is regarded as the dual type.
4. If the degree of Fidelity of Gestalt is low and the degree Functional Fidelity of the mental model is high, the mental model is assigned to the functional type.

The combination of the four aforementioned types in a coordinate system results in a diagrammatic representation (see Fig. 1).

A distinctive feature of this process is that a student is not required to go through all types when developing a mental model [20]. In other words, even though the presented

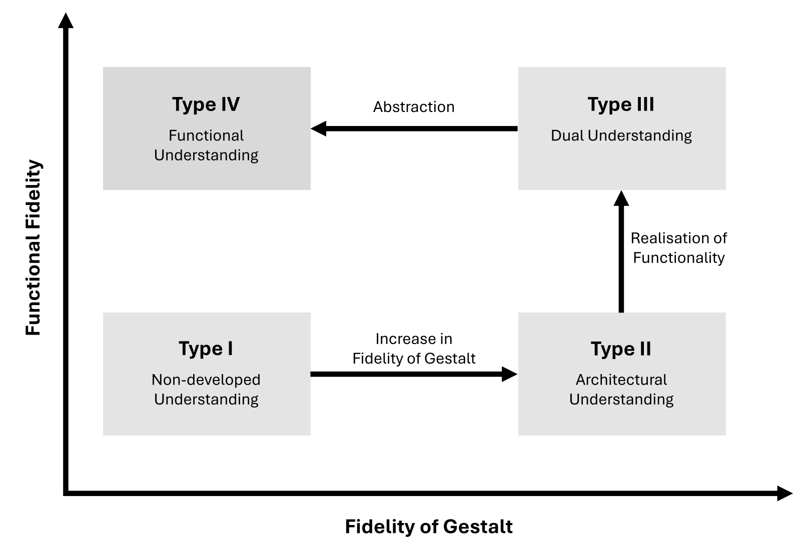

Figure 1 The combination of the four aforementioned types in a coordinate system [20, p.6]

structure is of hierarchical nature, learners do not need to move consecutively throughout all four types. However, as propsed by Ubben and Bitzenbauer the increase in the degree of Functional Fidelity is associated with a process of abstraction [20]. Mathematics, with its triad of definition, theorem, and proof, is a discipline that functions independently of any reality. Therefore mathematics can be used to describe the underlying reality in an abstract manner, which is why the introduction of mathematical notation might be conducive to increasing the degree of Functional Fidelity in learners' mental model.

2.2 Mathematics and its role in (quantum-)physics education

Mathematics plays a unique and integral role in the field of physics [25]. When investigating a physical phenomenon, a physicist may choose to rely on empirical observations or utilize mathematical models in order to gain insight into the underlying reality [26]. The latter is referred *mathematical sense making*, which entails the combination of conceptual understanding with the utilization and interpretation of a formal symbolic language [27]. In the *categorial sense making framework*, Gifford and Finkelstein propose four *reasoning structures* that describe how mathematical or physical contents are accessed [28]. An overview and a description of the abbreviations of those reasoning structures is provided in Fig. 2. All structures have in common that mathematical tools or physical models (e.g. photon model of light) serve as a mediator to enable learners to access the mathematical or physical object under investigation [28].

The categorial sense making framework furthermore describes two fundamental processes for knowledge acquisition: Firstly, the process of utilizing an already understood object as a foundation for understanding a subsequent object is referred to as *chaining* [28]. This connection is visualized with an green arrow in Fig. 3. Secondly, the concept of *coordination* refers to the interconnection between two reasoning structures that investigate the same object [28]. A visualization of both processes is provided in Fig. 3.

In instances where mathematical tools are employed during physics lessons, mostly "a mathematical formalism is leveraged to understand the behavior of a physical system" [28,

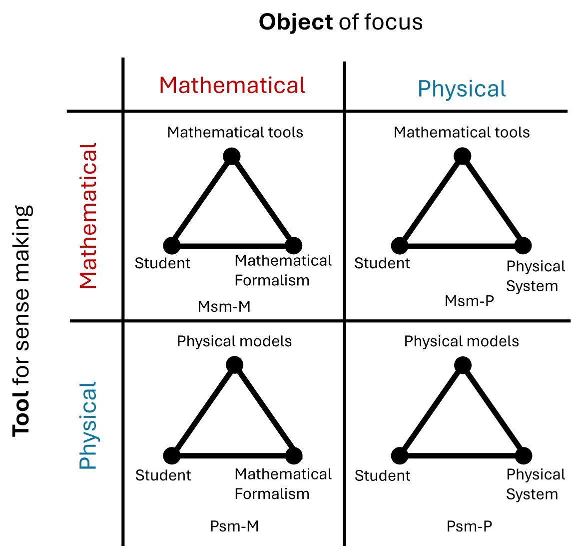

Figure 2 A visual illustration of the reasoning structures [28, p.4]. The following abbreviations are used: Msm-M: Mathematical sense making on a mathematical object; Msm-P: Mathematical sense making on a physical object; Psm-M: Physical sense making on a mathematical object; Psm-P: Physical sense making on a physical object;

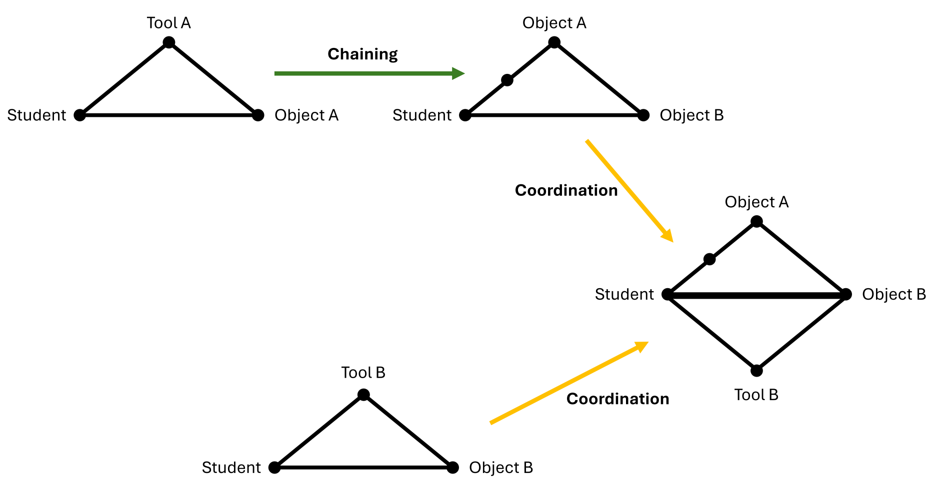

Figure 3 Connection of *reasoning structures*

p. 5]. A mathematical formalism is an indispensable tool in the field of QP, as a visualization of a single quantum object is inadequate for comprehending the underlying principles of quantum phenomena: "The interplay between mathematics and physics plays a special role in quantum physics. While in classical physics, mathematical formalism is usually derived from other types of representations (diagrammatical, pictorial), in quantum physics, the mathematics takes on the conceptual role" [29, p.88]. A multitude of distinct formulations of QP have been proposed, all of which are mathematically equivalent [30]. In the case of secondary school, many approaches to the mathematical formulation of QP employ Dirac notation [31], which may have an positive impact on the formal description of QP [32]. The great advantage of the Dirac notation is that instead of focusing on the mathematical representation of states (e.g., representing them by vectors or functions), it aims to capture the experimental depth of quantum physics using only some basic algebra [33]. In Dirac notation, quantum states are represented by abstract mathematical objects, so-called "kets" and quantum states can be expressed as a linear combination of "kets". Since "kets provide an elemental representation of basic states" [34], and the probability amplitudes are readily ascertainable, "in Dirac notations, it [the notation itself] became clear and meaningful" [35, p. 7].

Nevertheless, a considerable number of students continue to experience challenges with Dirac notation, even after receiving instruction [36–38]. Among the challenges identified were difficulties in applying scalar products to determine probabilities [38] and the transformation of a ket vector into the associated wave function [36, 37]. An approach that possibly circumvents these learning difficulties is provided by the use of a reduced Dirac notation. In a reduced Dirac notation, the scalar products are not calculated and therefore bra vectors must not be introduced.

The ket vectors function as a symbolic representation of a basis state [2, 31, 39]. In addition, a reduced Dirac notation enables the description of two-state systems [2, 39–41], which provide a foundation for understanding the essential concepts of quantum technologies in physics education [41–44]. This approach is exemplified by Scholz et al., whose instruction on quantum optics concentrates on quantum optical effects that are not adequately explained by models of either light beams or light waves – rather, they can be understood exclusively within the context of the photon model for light [39]. In this instructional sequence, the authors present a reduced Dirac notation that accurately describes the results of QP experiments with single photons used in their instructional sequence [39]. However, the fact that matrix multplication is not adressed in school mathematics renders this formalism unsuitable for use in secondary schools.

An alternative approach is proposed by Hennig et al. [2], whose instructions are based on those of Scholz et al. [39] mentioned above. In the instructional elements, experiments with single photons, presented via interactive screen experiments [45], as well as GeoGebra-animations, are employed as an instructional tool. In addition to a qualitative description and interpretation of the experiments, from which the characteristics of a single photon can be derived, Hennig et al. [2] introduce a reduced Dirac notation. In this notation, the phases of the basic states are represented with the help of arrows, and the probabilities are derived from the ratio of the magnitude squares of the individual arrow lengths [2]. This reduced Dirac notation does not require knowledge of either the elements of vector calculus or complex numbers. Consequently, it is accessible to learners with a less solid foundation in mathematics [2]. For a more detailed description, we refer the reader to our earlier work [2], as well as to Sect. 4.2. The fundamental idea of the learning path is that the sensemaking of the properties of a single photon occurs in accordance with the process illustrated in Fig. 4.

Various studies have been conducted to investigate the ways in which students learn the challenging topics of QP through the use of mathematics, as well as the difficulties they encounter in doing so [36, 37, 46, 47]. Indeed, there is empirical evidence that such teaching approaches, which suggest supplementing the qualitative treatment of QP concepts with a formalistic description of the corresponding phenomena appropriate to the target group, can be particularly conducive to learning. For example, Justice et al. [48] conducted a field study in which the experimental group, which received a hybrid instruction consisting of

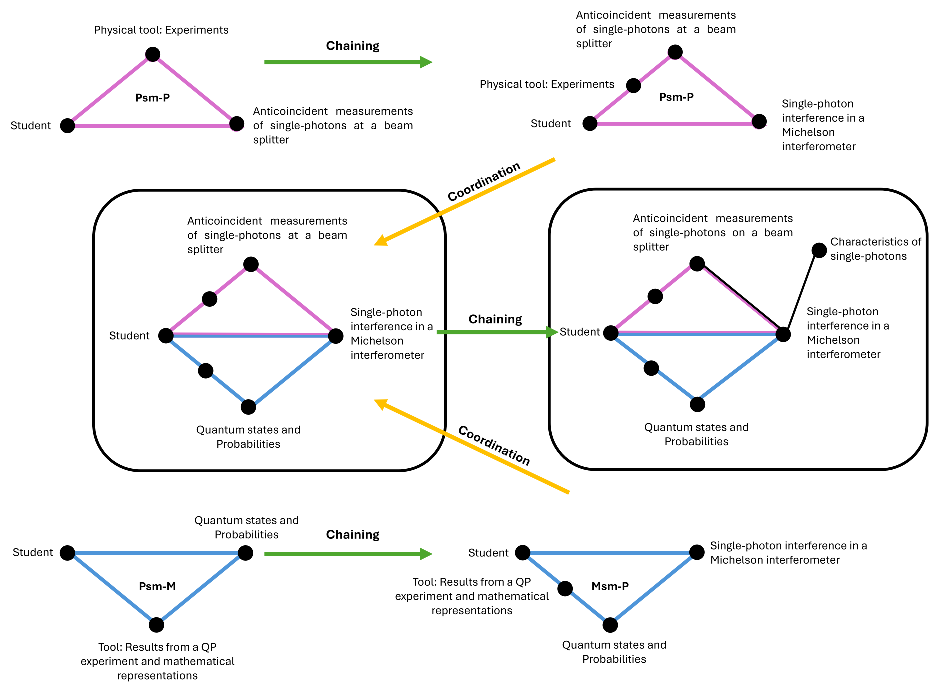

Figure 4 The figure depicts the sense making process that occurs throughout the learning path. The characteristics of photons can be deduced qualitatively (purple) or enhanced through quantitative reasoning (blue). In the initial qualitativ step, the behavior is determined with the assistance of a physical experiment. In the context of mathematical sensemaking, it is essential to first establish a reduced Dirac notation. In both the qualitative and formalistic reasoning paths, the findings result in the formulation of novel explanations for observable phenomena. Subsequently, the two reasoning structures can be coordinated. The objective of this course of study is to gain an understanding of the characteristics of individual photons

a qualitative and a formalistic part, demonstrated superior performance on a concept test relative to the control group, which solely received a qualitative instruction. However, a second experimental group exhibited inferior performance relative to the control group. In light of these findings, it remains uncertain whether the introduction of a mathematical formalism enhances the functionality of mental models. Consequently, additional investigation into this matter is required. Furthermore, it would be beneficial to examine the impact of employing a mathematical formalism on the working memory of learners, given that they are simultaneously engaged in learning physics and utilizing a mathematical tool. In physics instruction, the process of mathematical sense making with regard to a given physical system is often challenging [28]. It is not always the case that the behaviour of a physical system can be discerned directly with the aid of a mathematical model; in some instances the sense making process is "requiring Msm-M type [see Fig. 2] reasoning to understand the formalism before it can be used to understand physical behavior" [28, p.5]. Even if a formalism is deemed to be comprehended by a learner, it will nevertheless occupy cognitive resources when applied to a quantum physical situation to elucidate the underlying concepts, thereby precluding the limited working memory of a human from being available for other tasks [49–51]. It can be reasonably inferred that working memory is likely to be more heavily loaded in a setting in which a qualitative treatment is extended by a formalistic description. This leads to the question of whether learning is in fact more effective at the formalistic level, or whether the corresponding formalistic arguments have the effect of impeding the development of concepts in learners.

2.3 Cognitive load

The cognitive load theory by Sweller et al. [49, 52] assumes a limited working memory, which is influenced by a cognitiv load. In a modern view, there are two types of cognitive load which are determined by the structure of the learning environment or the learning tasks [52, 53]:

1. The intrinsic cognitive load "is concerned with the natural complexity of information that must be understood and material that must be learned, unencumbered by instructional issues such as how the information should be presented or in what activities learners should engage to maximise learning" [53, p.124].
2. The extraneous cognitive load (EL) is caused by "nonoptimal instructional procedures" [53, p.125].

In order to ascertain the degree of the various types of cognitive load experienced by learners, a series of test instruments have been developed [54, 55] and employed [56–58]. To quantify cognitive load the individual types – intrinsic load (IL) and extinsic load (EL) – were measured separately.

In the existing literature, there has been a limited focus on the cognitive load (CL) of students engaged in learning QP through the lens of mathematical representations (for an example, see [59]). In this paper, we revisit the research desiderata identified in Sect. 2.2 and Sect. 2.3 and conduct a clusterrandomized field studies to investigate the extent to which the reduced Dirac notation described by Hennig et al. [2] supports students in promoting Functional Fidelity in their thinking about photons with respect to the cognitive load they experienced.

3 Research questions

As mathematical descriptions of physics are often far less amenable to gestalt-like interpretations, it has been suggested in the literature that mathematical formalism could prove an influential factor in the advancement of functional understanding in the field of QP [20]. Concurrently, the implementation of a mathematical formalism may present two potential challenges. Firstly, learners may demonstrate proficiency in problem-solving but lack a comprehensive understanding of the underlying concepts [60]. Secondly, learners may experience cognitive overload, which could impede the promotion of Functional Fidelity in the thinking of students. This clusterrandomized study employs the teaching concept on the QP of the photon at the secondary level to empirically examine the extent to which a mathematical description can facilitate Functional Fidelity in the thinking of students and elucidate the role of cognitive load in this process. This gives rise to the following research questions (RQs):

- RQ 1 To what extent can the introduction of a mathematical formalism of QP making use of Dirac notation help students develop a functional understanding of photons?
- RQ 2 How does the (intrinsic and extrensic) cognitive load that students have to bear when they are introduced to formalistic descriptions of the outcomes of quantum experiments using the reduced Dirac notation compare to that of students who are introduced only to qualitative explanations of the respective experiments?

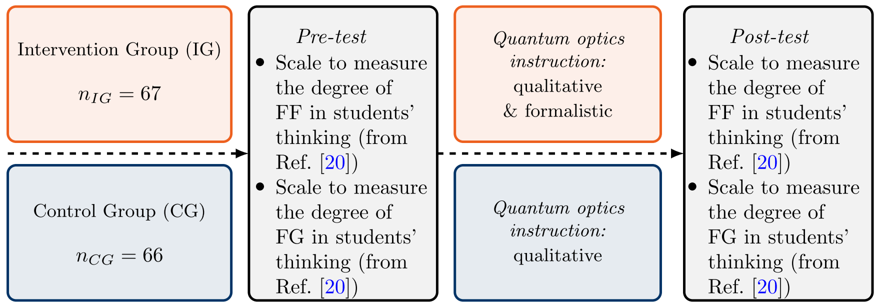

Figure 5 Overview of the study design. The instruction took a total of five school lessons (à 45 mins) and a detailed description of the interventions can be found in 4.2

Table 1 In the table is a summary of the study samples provided

	Total Sample	Intervention Group	Control Group
Students	133	67	66
Teachers	2	1	1
Classes	7	3	4

Two studies were conducted to address the research questions. The findings of the first study, which contributes to RQ 1, are presented in Sect. 4. Based on the results of this initial study, a follow-up study contributing to a clarification of RQ2 was conducted and the results of this study are reported in Sect. 5.

4 Study 1: clusterrandomized field study

4.1 Study design and sample

We conducted a clusterrandomized study with an intervention group (IC) and a control group (CG) in a pre-post test format, to examine the first RQ. The study was implemented in the field, i.e., in the classroom setting and during the regular physics lessons of the participating classes. An overview of the sample is given in Table 1 and an overview of the study design is provided in Fig. 5. The IC received instructions in strict accordance with the teaching-learning sequence presented in Ref. [2], wherein a qualitative treatment of single-photon experiments is enhanced by a formalistic description of single-photon states based on a reduced Dirac notation. The control group underwent the same instructions, but without the formalistic (i.e., only quantitative reasoning was used) description of the quantum-physical phenomena. The instructions for the IC and the CG each comprised five lessons.

The study was conducted with a sample of students aged between 15 and 17 years, with a total sample size of $n_{tot} = 133$.

4.2 Interventions

This section presents a description of the learning sequence experienced by the intervention group. The Table 3 at the end of the section provides a comparative summary of the learning sequences. The GeoGebra animations and worksheets utilized in this study can be accessed on the website.[1]

[1] https://fiztan.phd.elte.hu/letolt/erlangen-qm/.

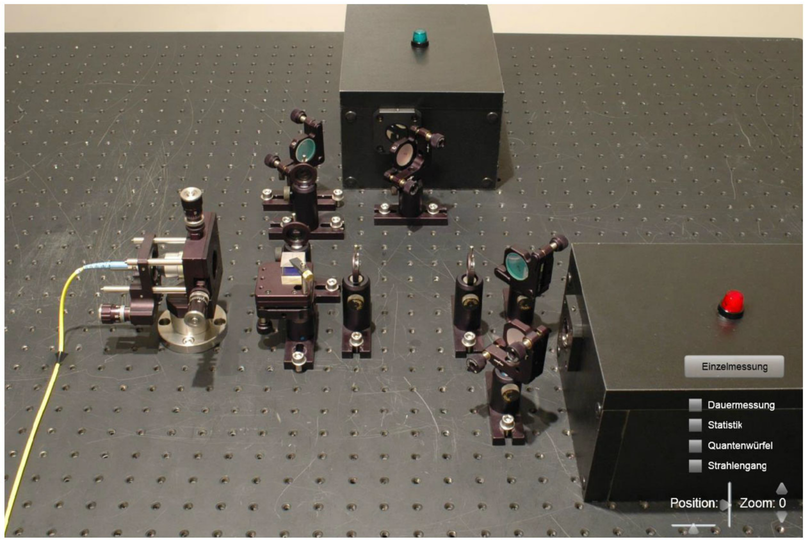

Figure 6 Interactive screen experiment on the anticorelation of the measurementresults of single-photon states at a beam splitter [45]

In the first lesson, selected topics regarding electromagnetic waves are repeated. This lesson therefore made it possible to bring the prior knowledge of the two groups (IG and CG) to a similar level. The students are instructed on the function of the beam splitter as an optical component and observe the behaviour of laser light at a 50:50 beam splitter in a demonstration experiment. Moreover, selected properties of electromagnetic waves, such as phase and interference, are examined in detail. Learners are presented with a conceptual link between the distinct phases of electromagnetic waves and the phenomenon of wave interference. During the lesson a GeoGebra animation, accessible via the link in the footnotes,[2] is employed to illustrate the phase shift that occurs during reflection at the beam splitter [61]. In the following lesson, the behavior of a single photon at the beam splitter will be investigated through the use of an interactive screen experiment (see Fig. 6) by Bronner et al. [45]. In the experimental setup, the lamps on the black boxes are illuminated when the detector in the box detects a single photon. The red (or green) lamp lights up when a single photon is found in the spatial position corresponding to the transmission (or reflection) at the beam splitter, respectively.

During the lesson, students can observe that both lamps never remain illumination at the same time. This experiment demonstrates the anti-correlation of a repeated measurements of a single photon. Three properties of photons can be derived from the observations made in the experiment.

1. Photons are indivisible and indistinguishable energy portions of light. Since the photons are completely identical (and do not interact with each other), a repeated measurement with a single photon prepared in the same state is completely identical to a single measurement of an ensemble consisting of many photons prepared in the same state.
2. A single photon is measured either by the detector D_T or by the detector D_R.

[2] https://www.geogebra.org/m/jmxhck52.

3. It is not possible to make any predictions regarding a single measurement results. But "statistical predictions (for many repetitions) are possible" [62, p. 3].

In accordance with the framework proposed by Gifford and Finkelstein [28], the experiment is utilized as a tool to comprehend the characteristics of photons. Given that a physical experiment was selected as the conduit for this knowledge process, the reasoning structure is from the type Psm-P (see Fig. 2).

Observations are necessary for the conceptualization of *quantization* and *probabilistic nature of a single quantum measurement.* These allow the introduction of the concept of superposition as a reason for probabilities. Probabilities must be used because the possible states of photons are not only the measured states (transmission or reflection), but also the so-called superposition states too (linear combination of them) [10]. It is not possible to make any predictions regarding a single measurement result, as a single photon is in a "superposition of transmitted or reflected" [62, p.4].

Following an initial overview of the concept of a state in classical mechanics, the subsequent section turns to an investigation of the quantum physical state and its distinctive characteristics. The QP allows the preparation of superposition states. In this learning sequence, we utilized a reduced Dirac notation to describe quantum states. This representation expressed the state of a quantum object using a so-called "Ket", for example the state of a photon is written as $|\psi\rangle$. In the experimental setup illustrated in Fig. 6, the state of a single photon can be represented as $|\psi\rangle = \varphi_{0^\circ}|S\rangle$. The ket $|S\rangle$ indicates that the photon has been emitted. The symbol φ_{0° represents the phase coefficient of the quantum state, which is similar to the phase of an electromagnetic wave. The phase coefficient can be represented by an arrow whose tip is oriented away from the coordinate origin. This interpretation of the probability amplitudes circumvents the necessity of introducing complex numbers and the necessity of calculating scalar products for normalization purposes.

Due to the placed beam splitter, the state of the photons must be considered as a superposition: $|\psi\rangle = \varphi_{90^\circ}|R\rangle + \varphi_{0^\circ}|T\rangle$. At this point in the learning path, students revisit the preceding results. The notation does not imply that a single photon is simultaneously in the states $|R\rangle$ and $|T\rangle$, because a single photon is "undividable". Rather, it is in a distinct state, known as a superposition state. Furthermore, a phase shift of 90° is observed during reflection at the beam splitter. At this point, the students can feel that the phenomenology of QP is difficult to visualize, whereas mathematics offers a simple tool to describe the experiments that are consistent with the measurements outcomes.

At last, the frequencies of the individual detections are elucidated in detail. If the interactive screen experiment is conducted for a sufficient duration, the quotient $\frac{N_{ref}}{N_{trans}}$ will approaches 1. The statistical determinism [63] leads to the conclusion that

$$\frac{N_{ref}/N}{N_{trans}/N} \to \frac{p(R)}{p(T)} = 1.$$

Given that the equation $p(R) + p(T) = 1$ is also a necessary condition, it can be concluded that $p(R) = p(T) = 0.5$ is the only solution of the equation system. Since we used arrows to represent the probability amplitudes in our notation, we found the ratio-equation $\frac{p(R)}{p(T)} = \frac{|\varphi_{90^\circ}|^2}{|\varphi_{0^\circ}|^2}$. This enabled us to connect the phase coefficients and the detection probabilities.

It is important to note that the quantum states are not normalized in this reduced Dirac notation, since the scalar product of states are not introduced at all. According to the

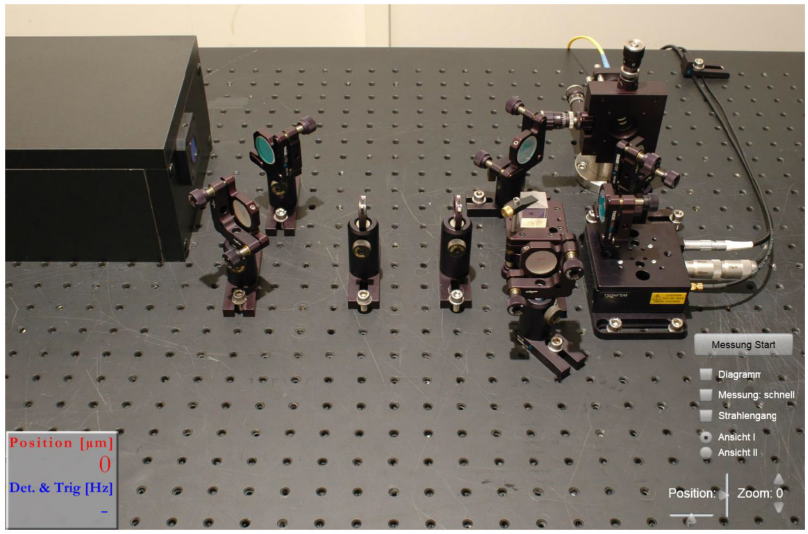

Figure 7 Interactive screen experiment on the phenomenon of single-photon interference in a Michelson interferometer [45]

framework of Gifford and Finkelstein [28] the sensemaking structure of the lesson is a Psm-M reasoning structure (see Fig. 2), whereby a mathematical formalism is rendered plausible through the aid of a physical model. To gain further insights into the characteristics of a single photon, an interactive screen experiment (see Fig. 7) by Bronner et al. [45] on single-photon interference in a Michelson interferometer is used in the fourth lesson.

Students can observe that in this interactive screen experiment, the detection rate, which is defined as the number of photons measured per second, is dependent on the position of the movable mirror of the Michelson interferometer. It is impossible to explain these experiences in the classical particle picture. The observation of single-photon interference can be attributed to the principles of delocalization and superposition. In conjunction with the students, a reflection on the observations yielded from the two screen experiments will be conducted through the reading of an excerpt from a research paper by Grangier et al. [64]. The authors of this article conclude that the experiments "illustrate the wave-particle duality of light. Indeed, if we want to use classical concepts, or pictures, to interpret these experiments, we must use a particle picture for the first one ('the photons are not split on a beam splitter') [...]. On the contrary, we are compelled to use a wave picture ('the electromagnetic field is coherently split on a beam splitter') to interpret the second (interference) experiment" [64, p. 178]. Based on this text excerpt, it is obvious that photons can neither be an electromagnetic wave nor a classical particle, and therefore single-photons are so-called quantum objects that obey the laws of QP.

Within the framework proposed by Gifford and Finkelstein [28], this connection of reasoning structures is called *chaining* (see Fig. 3), whereby further characteristics of a single photon can be accessed using the already known characteristics of them. The reasoning is from the type Psm-P, because we use a physical model to comprehend the single-photon interference (see Fig. 2).

In the last lesson, we use the reduced Dirac notation to describe single-photon interference in a formalistic way (see Table 2). For a comprehensive presentation of the argumentation, we refer the reader to the work of Hennig et al. [2]. According to the framework

Table 2 Heuristic reasoning for the formalistic description of single-photon states in a Michelson interferometer. This part of the teaching-learning sequence enables the "coordination" between the mathematical reasoning and its physical interpretation concerning single-photon interference in a Michelson interferometer

Description	Description of single-photon states and Calculations
A single photon is emitted from the single-photon source.	$\lvert\psi\rangle = \varphi_{0^\circ}\lvert S\rangle$
The quantum state of a single photon was changed by the beam splitter.	$\lvert\psi\rangle = \varphi_{90^\circ}\lvert R\rangle + \varphi_{0^\circ}\lvert T\rangle$
The effect of the mirrors on the quantum state.	$\lvert\psi\rangle = \varphi_{90^\circ}\lvert R\rangle + \varphi_{x^\circ}\lvert T\rangle$
The state of a single photon was changed by the beam splitter.	$\lvert\psi\rangle = \varphi_{90^\circ}\lvert D\rangle + \varphi_{180^\circ}\lvert S\rangle + \varphi_{90+x^\circ}\lvert D\rangle + \varphi_{x^\circ}\lvert S\rangle$
Output state	$\lvert\psi\rangle = (\varphi_{90^\circ} + \varphi_{90+x^\circ})\lvert D\rangle + (\varphi_{180^\circ} + \varphi_{x^\circ})\lvert S\rangle$
Application of $\frac{p(A)}{p(B)} = \frac{\lvert\alpha\rvert^2}{\lvert\beta\rvert^2}$	$\frac{p(S)}{p(D)} = \frac{\lvert\varphi_{180^\circ}+\varphi_{x^\circ}\rvert^2}{\lvert\varphi_{90^\circ}+\varphi_{90+x^\circ}\rvert^2}$
Pythagorean theorem & trig. identities	$\lvert\varphi_{180^\circ} + \varphi_{x^\circ}\rvert^2 = 4\sin^2\left(\frac{x}{2}\right)$ and $\lvert\varphi_{90^\circ} + \varphi_{90+x^\circ}\rvert^2 = 4\cos^2\left(\frac{x}{2}\right)$
Condition $p(A) + p(B) = 1$	$p(D) = \cos^2\left(\frac{x}{2}\right)$ and $p(S) = \sin^2\left(\frac{x}{2}\right)$
Using $x = \frac{360^\circ \cdot 2\Delta L}{\lambda}$	$p(D) = \cos^2\left(\frac{360^\circ \cdot \Delta L}{\lambda}\right)$ and $p(S) = \sin^2\left(\frac{360^\circ \cdot \Delta L}{\lambda}\right)$

Table 3 Comparison of the learning pathways for the intervention group (left) and the control group (right)

Lesson	Intervention group	Control group	Teaching Materials
1	Repetition on em-waves	Repetition on em-waves	GeoGebra Animations, Worksheets
2	Anticorrelation of photons at the beamsplitter	Anticorrelation of photons at the beamsplitter	Interactive screen experiment, Worksheet 1
3	Introduction of a reduced Dirac notation and reflection on preparation and measurement of quantum states	Qualitative reflection on preparation and measurement of quantum states	Worksheet 2
4	Single-photon interference in a Michelson interferometer	Single-photon interference in a Michelson interferometer	Interactive screen experiment, Worksheet 3
5	Reflection on the formalistic description of single-photon interference using the reduced Dirac notation	Qualitative reflection on single-photon interference	Worksheet 4

by Gifford and Finkelstein [28], in this lesson the so-called chaning is used to connect two reasoning structures 3. Furthermore, a Msm-P resoning structure is conducted in the lesson, whereby a mathematical formalism is used to comprehend further characteristics of a single photon (see Fig. 2).

In the lessons 3 and 5, in which the IC faces with the reduced Dirac notation, the CG engages in a qualitative reflection on the topics of quantum measurement and preparation of quantum states. A comparison of the two learning pathways for the IC and the CG is presented in Table 3.

4.3 Assessment of functional fidelity and fidelity of gestalt in students' mental models

To assess the degree of Functional Fidelity in learners' thinking about photons a scale (referred to as *FF scale* in the following) comprising seven 5-point rating-scale items was used and students had to rate the level of agreement with the statements provided (where 1 means "I do not agree at all" and 5 means "I completely agree"). The items were partly adopted from previous research [20], and three items were newly developed for this study. The list of all seven items can be found in Table 4). For the FF scale, Chronbach's alpha as an estimator for the scale's internal consistency was found to be $\alpha = 0.69$, which can be considered acceptable due to the shortness of the scale.

4.4 Data analysis

In order to investigate RQ 1, we determined the mean value and standard deviation (SD) for the Functional Fidelity scale. To investigate the differences in the degree of Functional Fidelity in the thinking of learners in the groups, a Mann-Whitney-U test was employed due to the non-normal distribution of the data. The Mann-Whitney-U test statistics are presented as follows: $U(n_{\mathrm{IC}}, n_{\mathrm{CG}}) = (U, z, p; r)$ where U represents the test statistic, n denotes the number of observations in each group, and p is the associated p-value. Furthermore z is the standardized test statistic. In addition, for statistically significant results we report the rank-biserial correlation

$$r = 1 - \frac{2U}{n_{\mathrm{IG}} \cdot n_{\mathrm{CG}}}$$

as a measure of effect size. For further details, we refer readers to refernce [65]. Statistical significance is defined as a p-value less than 0.05, while a p-value below 0.01 indicates a highly significant result. Following Kerby [66], we classify effect sizes with $0.2 < r < 0.4$ as medium and those with $r \geq 0.40$ as large. Furthermore, a Wilcoxon test was employed to examine whether there is a statistically significant increase in the degree of Functional Fidelity of the students' thinking. The Willcoxon test statistics are presented as (z, p).

4.5 Results of study 1

This section presents the individual results of the Functional Fidelity Scale for both the experimental and control group. In particular, the median, mean, and standard deviation (SD) of student ratings for each item at both the pre-test and post-test points in time were presented in tabular form (see Table 4). Furthermore the median, mean, and standard deviation (SD) of the overall student ratings on the Functional Fidelity scale were presented for both the pre-test and post-test point in time (see Table 5).

The data indicated that, prior to the intervention, no statistically significant differences in the degree of Functional Fidelity in learners' thinking could be identified ($U(67) = 1948$, $z = -1.19$, $p = 0.24$). A Wilcoxon test revealed a statistically significant difference in the degree of functional thinking prior and post instruction in both the IC ($z = 4.67$, $p < 0.01$) and CG ($z = 7.06$, $p < 0.01$). Moreover, the degree of Functional Fidelity in the thinking of students from both groups differs significantly ($U(67) = 1542$, $z = -3.02$, $p < 0.01$; $r = 0.30$), and students in the CG demonstrated a stronger tendency towards functional thinking than those in the IC.

4.6 Discussion of RQ 1

On the one hand, we positively evaluate that the learning path presented in the study by Hennig et al. [2] significantly improves the functional thinking about photons. Our results highlight that teaching QP in high schools via the teaching-learning path suggested by Hennig et al. [2] is indeed worthwhile.

Table 4 Overview of the descriptive statistics regarding the rating-scale items from the functional fidelity scale. The questions were related to two (thought) experimental situations. In the first one, a single photon is emitted onto a beam splitter, in the second one, a single photon was considered in a Michelson interferometer. The students' answers were coded as follows: 1 = I do not agree at all, 2 = I rather do not agree, 3 = I am not sure, 4 = I rather agree, 5 = I completely agree

Item	Mean				SD			
	Pre		Post		Pre		Post	
	IC	CG	IG	CG	IG	CG	IG	CG
For the next statements, consider an experiment in which a single photon is emitted onto a beam splitter and two detectors are placed behind it according to the transmission and the reflection.								
1. The two detectors never sign simultaneously because a single photon cannot be divided.	3.24	2.81	4.52	4.95	1.28	1.06	0.98	0.21
2. It is not possible to predict with certainty which detector will find them.	3.16	2.73	4.03	4.42	1.22	1.02	1.24	1.03
3. The accurate position of a single photon at any given time between the beam splitter and the detectors cannot be determined in principle.	2.81	2.81	3.69	4.38	1.01	0.89	1.22	0.93
4. It is feasible for a single photon to have no classically well-defined position.	3.13	3.14	3.79	4.55	0.94	1.06	1.09	0.86
For the next statements, consider a single photon in a Michelson interferometer. A single-photon source is used, so only a single photon is emitted in a given time.								
5. The beam splitter does not make a single photon split; however, a single photon has no certain trajectory between the beam splitter and detector.	2.84	2.71	3.82	4.28	1.00	1.01	1.15	0.90
6. A single photon cannot be split by the beam splitter, but an interference pattern can be seen when the position of the mirror M_T is constantly changing.	3.19	3.47	4.01	3.86	1.12	0.89	1.09	0.97
7. Using the interference pattern, we could not calculate the certain trajectory of a single photon.	2.84	2.94	3.61	3.92	1.06	0.87	1.33	1.02

Table 5 Descriptive statistics on pre-test and post-test scores for the Functional Fidelity scale for each group

	Group	Median	Mean	SD	Mann-Whitney-U	*p*	*r*
Pre-test	CG	2.93	2.95	0.43	$U(67) = 1948$	0.24	
	IC	3.14	3.03	0.54	$z = -1.19$		
Post-test	CG	4.43	4.34	0.46	$U(67) = 1542$	< 0.01	0.30
	IG	4.00	3.96	0.72	$z = -3.02$		

Before the teaching experiment, the functionality of the mental models in both groups was at a similar level (see Fig. 8) no statistically significant difference was measured, and even most individual questions showed similar values (see Table 4). However, the post-test results yielded that the Intervention Group (IG), which was supplemented with mathematical formalism of QP, exhibited a highly statistically significant underperformance compared to the Control Group (CG). There was only one question (item number 6, i.e., in the context of a single-photon Michelson interferometer *"A single photon cannot be split by the beam splitter, but an interference pattern can be seen when the position of the mirror M_T is constantly changing."*), in which the IG outperformed the CG (M_{IG} = 4.01 and M_{CG} = 3.86), despite the fact that the CG had better pre-test score. The quantum interference of a single photon is visually unobservable, and therefore, qualitative explanations are necessarily imprecise; only the mathematical formalism can provide an accurate description of the phenomenon. The fact that the IG outperformed the CG in this purely quantum phenomenon indicates that the mathematical description was particularly useful in this context. Our results indicate that the learning path supplemented with a mathematical

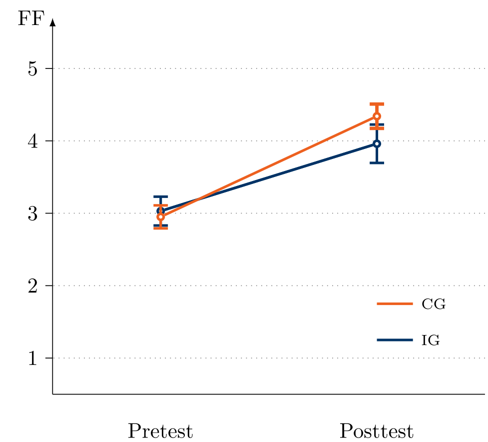

Figure 8 Change in average Functional Fidelity score FF within intervention group (IG) and the control group (CG). The error bars indicate the 95% confidence interval

formalism is less effective than the purely qualitative discussion in the development of the students' functional mental models, and that the formalism provided help only in the question related to quantum interference. It is important to note that both groups spent the same amount of time learning the presented material. Therefore, the mathematical formalism appeared as an additional element during the lessons for the IG.

Our findings, therefore, contextualize the research results of Pospiech et al. [32], which reported on the potential positive impact of incorporating Dirac notation into the mathematical formalism of QP on the learning process of high school students. It appears, however, that the pedagogical application of mathematical formalism in QP is not always the best choice. This is consistent with the findings of Justice et al. [48], who found that while the mathematical formalism facilitates the learning process for upper-level undergraduate students, mixed results were observed among graduate students. They attempted to explain their results by attributing them to the cognitive overload experienced by students due to the formal descriptions.

Our answer to the first research question (RQ1) is therefore: Under the learning conditions outlined above, although the mathematics formalism in QP using the reduced Dirac notation helps students' functional understanding of photons, students who did not gain insight into the mathematical formalism still performed better in functional understanding. Since quantum formalism is fundamentally expected to play a clarifying role, we feel it is important to supplement our research with an examination of the cognitive load imposed by the mathematical formalism on the students. Our hypothesis is that the mathematical formalism may have presented an additional burden during the lessons, thereby hindering the students' proper learning process. Thus, in the next chapter, we present a follow-up study focusing on cognitive load.

5 Study 2: a follow-up

The results of the preceding study substantiated the hypothesis that the introduction of a mathematical formalism enhances the degree of Functional Fidelity of learners' thinking. It has been observed that CG students tend more to functional thinking than the students of the IC, which gives rise to the question of what the potential causes of this result may be.

Table 6 In the table is a summary of the study samples provided

	Total Sample	Intervention Group	Control Group
Students	136	71	65
Teachers	2	1	1
Classes	7	4	3

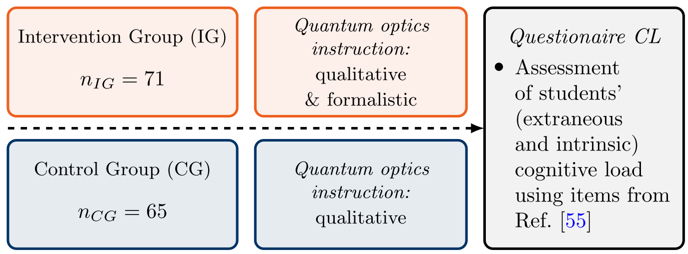

Figure 9 Overview of the study design

As learning processes are associated with a cognitive load (see Sect. 2.2), an investigation of the cognitive load may provide a rationale for the findings of Study 1.

5.1 Study design and sample

To examine the cognitive load, a clusterrandomized field study was conducted with a sample of n_{tot} = 136 students from seven german classes. Four classes, comprising a total of n_{IC} = 71 students, received the same instruction as the IC from study 1 (see Sect. 4.2). The three remaining classes, with a total of n_{CG} = 65 students, underwent the same learning sequence as the CG from study 1. The Table 6 provide an overview of the sample of the study.

In order to circumvent the measurement of the CL during the processing of the FF scale, we chose not to repeat a measurement of the degree of the Functional Fidelity in learners' thinking. Therefore the study was conducted as shown in Fig. 9.

5.2 Assessment of cognitive load

The CL is constituted of two distinct types, namely the intrinsic cognitive load (IL) and extraneous cognitive load (EL) [53]. To assess both IL and EL, we employed scales that have been previously developed and utilized in research by Leppink et al. [55]. The 11-level rating scale items were translated into german and underwent a contextual adaptation where necessary. Chronbach's alpha as a measure of internal consistency is $\alpha = 0.83$ for the IL scale and $\alpha = 0.82$ for the EL scale, and therefore the individual scales are considered as reliable.

5.3 Data analysis

In order to investigate RQ 2, we determined the mean value and standard deviation (SD) for each scale. As in the previous study, we used Mann-Whitney-U tests to examine differences in the IL and the EL between the two comparison groups for statistical significance. Analogeous to Sect. 4.5 the Mann-Whitney-U test statistics was presented as follows: $U(n_{\mathrm{IG}}, n_{\mathrm{CG}}) = (U, z, p; r)$.

Table 7 Overview of the descriptive statistics regarding the rating-scale items from the IL and the EL scale [55]. The students' answers were coded as follows: 0 = not at all the case,…, 10 = completely the case

Type	Item	Median		Mean		SD	
		IG	CG	IG	CG	IG	CG
IL	The topics covered in the activity were very complex.	8	6	7.25	6.24	2.25	1.66
IL	The activity covered formulas that I perceived as very complex.	5	8	7.69	5.46	2.02	2.25
IL	The activity covered concepts and definitions that I perceived as very complex.	8	6	7.32	5.86	2.01	1.75
EL	The instructions and explanations during the activity were very unclear.	6	5	6.09	5.45	2.40	2.53
EL	The instructions and explanations were, in terms of learning, very ineffective.	4	5	4.46	4.80	2.30	2.21
EL	The instructions and/or explanations were full of unclear language.	4	4	4.71	4.07	2.68	2.30

Table 8 Descriptive statistics on intrinsic and extraneous scores for IC and CG

	Group	Median	Mean	SD	Mann-Whitney-U	p	r
Intrinsic load	CG	5.67	5.85	1.57	$U(71) = 1164$	< 0.01	0.50
	IC	7.67	7.42	1.86	$z = -4.98$		
Extraneous load	CG	4.67	4.77	2.02	$U(71) = 2121$	0.417	
	IG	4.67	5.09	2.09	$z = -0.81$		

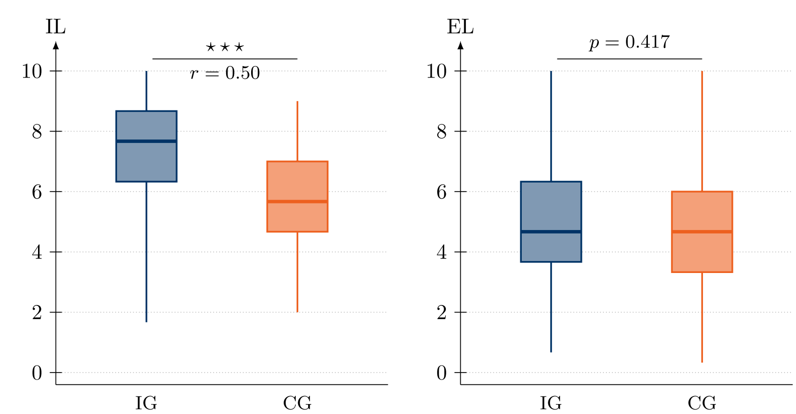

Figure 10 Boxplots for the intrinsic (left) and extraneous (right) cognitive loads within the intervention group (IG) and the control group (CG). The three asterisks highlight high statistical significance ($p < 0.01$) with regards to the Mann-Whitney-U test comparing both cohorts. The associated effect sizes r are displayed as well

5.4 Results of study 2

In this section we present the individual results of the IL and EL scales for the IC and the CG (see Table 7 and Fig. 10) as well as the results of the Mann-Whitney-U tests (see Table 8). The Table 7 summarizes the median, mean, and standard deviation (SD) of the two scales.

The data indicate that the IL resulting from the learning object is significantly higher in the IG than in the CG ($U(71) = 1164$, $z = -4.98$, $p < 0.01$; $r = 0.50$). This suggests that the IG experiencing a greater degree of IL relative to the CG, which can be attributed to the learning content. We will discuss this in more detail in Sect. 5.5. In contrast, the EL, which is influenced by external factors, does not exhibit a statistically significant difference between the two comparison groups ($U(71) = 2121$, $z = -0,81$, $p = 0.42$).

5.5 Discussion of RQ 2

Through statistical analysis we can give an answer to our second reseach question (RQ2). We confirmed our initial hypothesis that the mathematical formalism using the reduced Dirac notation statistically significantly demands more working memory (intrinsic load, IL), imposing a cognitive burden on students. This discrepancy was solely attributed to the mathematical formalism, as no statistically significant differences were found in the levels of extraneous cognitive load (EL) between the two groups [53]. Our research results are worth comparing with the relationship between mathematics and physics discussed in Sect. 2.2). It is possible that the higher IL in the intervention group (IG) was caused by the fact that some of the students were stuck in one of the learning processes. For example, an earlier manuscript [21] investigated the learning difficulties of students in the teaching-learning sequence proposed by Hennig et al. [2] and found that there were difficulties in some part of formal description, such as the physical interpretation of the mathematical object quantum phase or the formal description of quantum interference. Consequently, we infer four conclusions:

1. Based on Sect. 4, it would be helpful to look at the material from Hennig et al. [2] to see where the students got tripped up. For example, it is possible that some of the students were not able to grasp certain mathematical parts in sufficient depth.
2. Although mathematical formalism in QP facilitates the development of a functional QP mental model, attention must be given to students' levels of abstraction and mathematical background. The introduction of such formalism may result in a high intristic cognitive load, which could be less beneficial to the learning process.
3. For certain student groups, it may be beneficial to allocate more time for the introduction of mathematical formalism in QM, thereby reducing the intristic cognitive load.
4. It is also worth considering presenting the formalism in a simpler way to the students.

We therefore concur with the suggestion of Justice et al. [48] that the effectiveness of introducing quantum formalism in the learning process is highly dependent on the characteristics of the students group and the used material itself.

6 Limitations

We believe it is important to highlight the limitations of our research. The presented study is a clusterrandomized design, which inherently lacks randomization at the individual level. This limitation may lead to systematic differences between cohorts, particularly concerning variables not accounted for in this research. To address this, future experimental studies should include additional control variables, especially affective ones, to enhance the robustness of the findings.

The instrument used to measure Functional Fidelity was constrained by the time limitations of a real classroom setting, resulting in a brief scale. However, the Cronbach's Alpha value was relatively low (see Sect. 4.3), indicating that the questionnaire items could be operationalized more effectively. Improvements in the scale's design and the inclusion of more comprehensive items are recommended for future studies.

Regarding the study design, the division into two phases was a consequence of the initial phase's results (Study 1 in Sect. 4 and Study 2 in Sect. 5). Nevertheless, we were unable to collect linked data on Functional Fidelity scores and individual cognitive load for each student. Future research should replicate this study, utilizing ANCOVA with cognitive load variables as covariates in the data analysis to better account for these factors.

Lastly, this study focused solely on the impact of introducing a reduced Dirac notation on secondary school students' functional thinking about photons. While the results showed that these students were outperformed by those who only received qualitative instruction (Sect. 4.6) — likely due to the increased cognitive load associated with the mathematical content (Sect. 5.5) — it is important to consider that the formalistic approach to quantum phenomena may benefit other aspects of learning, such as students' perceptions of the nature of (quantum) science. Future research should explore these potential benefits. It is noteworthy that the participating students were 15-17 years old and did not attend classes specializing in physics; moreover, they were not particularly strong in mathematics. Additionally, both groups spent the same amount of time learning the presented material, so the quantum formalism appeared as an extension during the lessons (see Table 3). It is possible that with a group more experienced in physics and mathematics, we would have obtained the opposite result, that is, it is conceivable that the quantum formalism could much better facilitate the development of students' functional mental models. Furthermore, it is also possible that an extra learning time is required when the formalism is introduced or maybe there is a better way to introduce formalism.

Our study highlights a very important principle in curriculum design that teachers should take into account. In the teaching and learning of QP, students should have a prior knowledge of the mathematical foundations, and in addition, adequate time should be devoted to experimental and conceptual foundations in the classroom. Only then should mathematical description be introduced to prepared groups, allowing them sufficient time to digest it. Particular care should be taken to keep mathematical description as simple as possible, and to find a balance for our group of learners where their thinking benefits as much as possible from formal description, while intrinsic cognitive load remains moderate. In all of this, it is worth listening to and heeding the advice on the relationship between mathematics and physics given in the [28] study (Sect. 2.2).

7 Conclusion

This paper presents the findings of a clusterrandomized field study, which demonstrates that the implementation of a reduced Dirac notation can lead to a statistically significant enhancement of the degree of Functional Fidelity in learners' thinking about photons. This study, hence, provides important first insights into the potential efficacy of introducing a reduced Dirac notation to improve the degree of Functional Fidelity in learners' thinking about photons. Nevertheless, several research questions remain unresolved, warranting further investigation. Foremost, we suggest to conduct a replication of the study presented in this article ensuring to collect conceptual data and cogntive load data linked to each individual learner prior and post instruction such that the impact of cognitive load on student learning can better be accounted for. Also, collecting data on further covariates, such as the affective variables beyond cognitive load, seems valuable and would yield a more holistic understanding of the learning process facilitated by introducing a quantum formalism tailored to the secondary school level.

Furthermore, beyond quantitative data, further research is required to gain insight into (a) how and (b) the extent to which learners utilize the reduced Dirac notation when solving quantum physics problems. To this end, an exploratory study administering open-ended questions to learners post instruction seems a sensible approach.

Additionally, incorporating educators' perspectives could yield valuable insights into the feasibility and potential adoption of reduced Dirac notation in instructional practices. This would also provide an understanding of the methods and extent to which educators might integrate this notation into their teaching.

Author contributions
The study was designed and conducted by F.H., K.T. and P.B. The first version of the manuscript was written by F.H. and K.T, and J.V. and P.B. edited the manuscript. J.V. designed the figures and P.B. supervised the project.

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Data Availability
The data presented in this study are available on request from the corresponding author.

Declarations

Ethics approval and consent to participate
Ethical review and approval was not required for the study on human participants in accordance with the local legislation and institutional requirements.

Consent for publication
The participants provided their written informed consent to participate in this study.

Financial Interests
The authors have no relevant financial or non-financial interests to disclose.

Competing interests
The authors declare no competing interests.

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