Bouten et al. *EPJ Quantum Technology* (2015) 2:9
DOI 10.1140/epjqt/s40507-015-0024-2

EPJ.org
EPJ Quantum Technology
a SpringerOpen Journal

RESEARCH Open Access

A Trotter-Kato theorem for quantum Markov limits

Luc Bouten[1], Rolf Gohm[2*], John Gough[2] and Hendra Nurdin[3]

*Correspondence: rog@aber.ac.uk
[2]Dept. for Mathematics and Physics, Aberystwyth University, SY23 3BZ, Wales, United Kingdom
Full list of author information is available at the end of the article

Abstract
Using the Trotter-Kato theorem we prove the convergence of the unitary dynamics generated by an increasingly singular Hamiltonian in the case of a single field coupling. The limit dynamics is a quantum stochastic evolution of Hudson-Parthasarathy type, and we establish in the process a graph limit convergence of the pre-limit Hamiltonian operators to the Chebotarev-Gregoratti-von Waldenfels Hamiltonian generating the quantum Itō evolution.

1 Introduction

In the situation of regular perturbation theory, we typically have a Hamiltonian interaction of the form $H = H_0 + H_{\text{int}}$ with associated strongly continuous one-parameter unitary groups $U_0(t) = e^{-itH_0}$ (the free evolution) and $U(t) = e^{-itH}$ (the perturbed evolution), then we transform to the Dirac interaction picture by means of the unitary family $V(t) = U_0(-t)U(t)$. Although $V(\cdot)$ is strongly continuous, it does not form a one-parameter group but instead yields what is known as a left U_0-cocycle:

$$V(t+s) = U_0(s)^\dagger V(t) U_0(s) V(s). \tag{1}$$

One obtains the interaction picture dynamical equation

$$i\frac{d}{dt}V(t) = \Upsilon(t)V(t), \tag{2}$$

where $\Upsilon(t) = U_0(t)^\dagger H_{\text{int}} U_0(t)$.

More generally, we may have a pair of unitary groups $U(\cdot)$ and $U_0(\cdot)$ with Stone generators H and H_0 respectively, but where the intersection of the domains of the generators are not dense. This is the situation of a singular perturbation. In this case we cannot expect the Dirac picture dynamical equation (2) to be anything but formal since the difference $H_{\text{int}} = H - H_0$ is not densely defined.

Remarkably, the steps above can be reversed even for the situation of singular perturbations. If we assume at the outset a fixed free dynamics $U_0(\cdot)$, with Stone generator H_0, and a strongly continuous unitary left U_0-cocycle $V(\cdot)$, then $U(t) = U_0(t)V(t)$ will then form a strongly continuous one-parameter unitary group with Stone generator H. In practice however the problem of reconstructing H from the prescribed H_0 and $V(\cdot)$ will be difficult.

Springer

© 2015 Bouten et al.; licensee Springer. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

In the situation of quantum stochastic evolutions introduced by Hudson and Parthasarathy [1], we have a strongly continuous adapted process $V(\cdot)$ satisfying a quantum stochastic differential equation (including Wiener and Poisson noise as special commutative cases) in place of (2), and the solution constitutes a cocycle with respect to the time-shift maps $U_0 \equiv \Theta$ (see below). Nevertheless, $V(\cdot)$ arises as the Dirac picture evolution for a singular perturbation of a unitary $U(\cdot)$ with some generator H with respect to the time-shift: it was a long standing problem to find an explicit form for H which was finally resolved by Gregoratti [2], see also [3].

The purpose of this paper is to approximate the singular perturbation arising in quantum stochastic evolution models by a sequence of regular perturbation models. That is, to construct a sequence of Hamiltonians $H^{(k)} = H_0 + H^{(k)}_{\mathrm{int}}$ yielding a regular perturbation $V^{(k)}(\cdot)$ converging to a singular perturbation $V(\cdot)$ in some controlled way. We exploit the fact that the limit Hamiltonian is now known through the work of Chebotarev [4] and Gregoratti [2]. The strategy is to employ the Trotter-Kato theorem which guarantees strong uniform convergence of the unitaries once graph convergence of the Hamiltonians is established.

1.1 Quantum stochastic evolutions

The seminal work of Hudson and Parthasarathy [1] on quantum stochastic evolutions lead to explicit constructions of unitary adapted quantum stochastic processes V describing the open dynamical evolution of a system with a singular Boson field environment. We fix the system Hilbert space $\mathfrak{h}$ and model the environment as having n channels so that the underlying Fock space is $\mathfrak{F} = \Gamma(\mathbb{C}^n \otimes L^2(\mathbb{R}))$. Here $\Gamma(\mathfrak{H})$ denotes the symmetric (boson) Fock space over a one-particle space $\mathfrak{H}$: we set the inner product as $\langle \Psi | \Phi \rangle = \sum_{m=0}^{\infty} \frac{1}{m!} \langle \Psi_m | \Phi_m \rangle$ and take the exponential vectors to be defined as ($\otimes_s$ denoting a symmetric tensor product)

$$e(f) = (1, f, f \otimes_s f, f \otimes_s f \otimes_s f, \ldots)$$

with test function $f \in \mathfrak{H}$. Here the one particle space is $L^2(\mathbb{R})$, the space of complex-valued square-integrable functions on $\mathbb{R}$. We define the operators

$$\begin{aligned} \Lambda^{00}(t) &\triangleq t, \\ \Lambda^{10}(t) = A^\dagger(t) &\triangleq a^\dagger(1_{[0,t]}), \\ \Lambda^{01}(t) = A(t) &\triangleq a(1_{[0,t]}), \\ \Lambda^{11}(t) = \Lambda(t) &\triangleq d\Gamma(\chi_{[0,t]}), \end{aligned}$$

where $1_{[0,t]}$ is the characteristic function of the interval $[0,t]$ and $\chi_{[0,t]}$ is the operator on $L^2(\mathbb{R})$ corresponding to multiplication by $1_{[0,t]}$. Hudson and Parthasarathy [1] have developed a quantum Itō calculus where the basic objects are integrals of adapted processes with respect to the fundamental processes $\Lambda^{\alpha\beta}$. The quantum Itō table is then

$$d\Lambda^{\alpha\beta}(t)\, d\Lambda^{\mu\nu}(t) = \hat{\delta}_{\beta\mu}\, d\Lambda^{\alpha\nu}(t),$$

where $\hat{\delta}_{\alpha\beta}$ is the Evans-Hudson delta defined to equal unity if $\alpha = \beta = 1$ and zero otherwise. This may be written as

$$\begin{array}{c|cccc} \times & dA & d\Lambda & dA^\dagger & dt \\ \hline dA & 0 & dA & dt & 0 \\ d\Lambda & 0 & d\Lambda & dA & 0 \\ dA^\dagger & 0 & 0 & 0 & 0 \\ dt & 0 & 0 & 0 & 0 \end{array} \cdot$$

In particular, we have the following theorem [1].

Theorem 1 *There exists a unique solution $V(\cdot,\cdot)$ to the quantum stochastic differential equation*

$$V(t,s) = I + \int_s^t dG(\tau)V(\tau,s) \tag{3}$$

($t \geq s \geq 0$) *where*

$$dG(t) = G_{\alpha\beta} \otimes d\Lambda^{\alpha\beta}(t)$$

with $G_{\alpha\beta} \in \mathfrak{B}(\mathfrak{h})$. (We adopt the convention that we sum repeated Greek indices over the range 0, 1.)

In particular, set $V(t) = V(t,0)$ then we have the quantum stochastic differential equation $dV(t) = dG(t)V(t)$ which replaces the regular Dirac picture dynamical equation (2).

We refer to $\mathbf{G} = [G_{\alpha\beta}] \in \mathfrak{B}(\mathfrak{h} \oplus \mathfrak{h})$, as the *coefficient matrix*, and V as the left process generated by $\mathbf{G}$. The conditions for the process V to be unitary are that $\mathbf{G}$ takes the form, with respect to the decomposition $\mathfrak{h} \oplus \mathfrak{h}$,

$$\mathbf{G} = \begin{bmatrix} -\frac{1}{2}\mathsf{L}^\dagger\mathsf{L} - i\mathsf{H} & -\mathsf{L}^\dagger\mathsf{S} \\ \mathsf{L} & \mathsf{S} - I \end{bmatrix}, \tag{4}$$

where $\mathsf{S} \in \mathfrak{B}(\mathfrak{h})$ is a unitary, $\mathsf{L} \in \mathfrak{B}(\mathfrak{h})$ and $\mathsf{H} \in \mathfrak{B}(\mathfrak{h})$ is self-adjoint. We may write in more familiar notation [1]

$$dG(t) = \left(-\frac{1}{2}\mathsf{L}^\dagger\mathsf{L} - i\mathsf{H}\right) \otimes dt - \mathsf{L}^\dagger\mathsf{S} \otimes dA(t) + \mathsf{L} \otimes dA^\dagger(t) + (\mathsf{S} - I) \otimes d\Lambda(t).$$

We denote the shift map on $L^2(\mathbb{R})$ by θ_t, that is $(\theta_t)f(\cdot) = f(\cdot + t)$ and its second quantization as $\Theta_t = I \otimes \Gamma(\theta_t)$. It then turns out that $\Theta_\tau^\dagger V(t,s)\Theta_\tau = V(t+\tau, s+\tau)$ and so $V(t) = V(0,t)$ is a left unitary Θ-cocycle and that there must exist a self-adjoint operator H such that

$$\Theta_t V(t) \equiv e^{-iHt}$$

for $t \geq 0$. (For $t < 0$ one has $V(-t)^\dagger \Theta_{-t} \equiv e^{-iHt}$.) Here H will be a singular perturbation of generator of the shift, and its characterization was given by Gregoratti [2]. See also [5].

1.2 Physical motivation

As a precursor to and motivation for further approximations, we fix on a simple model of a quantum mechanical system coupled to a boson field reservoir R. In the Markov approximation we assume that the auto-correlation time of the field processes vanishes in the limit: this includes weak coupling (van Hove) and low density limits. The Hilbert space for the field is the Fock space $\mathcal{F}_R = \Gamma(\mathcal{H}_R^1)$ with one-particle space $\mathcal{H}_R^1 = L^2(\mathbb{R})$ taken as the momentum space. (For convenience we consider a one-dimensional situation because this is the setting studied in this paper but of course $\mathbb{R}^3$ is particularly relevant physically.) It is convenient to write annihilation operators formally as $A_R(g) = \int_{\mathbb{R}} g(p)^* a_p \, dp$ where $[a_p, a_{p'}^\dagger] = \delta(p - p')$.

In particular, let us fix a function $g \in L^2(\mathbb{R})$, and set

$$a(t,k) = \sqrt{k} \int e^{-i\omega(p)tk} g(p)^* a_p \, dp,$$

where $\omega = \omega(p)$ is a given function (determining the dispersion relation for the free quanta) and k is a dimensionless parameter rescaling time. We have the commutation relations

$$\bigl[a(s,k), a(t,k)^\dagger\bigr] = k\rho\bigl(k(t-s)\bigr),$$

where

$$\rho(\tau) \equiv \int \bigl|g(p)\bigr|^2 e^{i\omega(p)\tau} \, dp.$$

The limit $k \to \infty$ leads to singular commutation relations, and it is convenient to introduce smeared fields

$$A(\varphi,k) = \int \varphi(t)^* a(t,k) \, dt$$

in which case we have the two-point function (and define an operator C_k by)

$$\bigl[A(\varphi,k), A(\psi,k)^\dagger\bigr] = \int dt \, dt' \varphi(t)^* k\rho\bigl(k\bigl(t-t'\bigr)\bigr)\psi\bigl(t'\bigr) \equiv \langle\varphi|C_k\psi\rangle.$$

For ρ integrable, we expect

$$\lim_{k\to\infty} \bigl[A(\varphi,k), A(\psi,k)^\dagger\bigr] = \gamma \int dt \varphi(t)^* \psi(t),$$

where $\gamma = \int_{-\infty}^{\infty} \rho(\tau) \, d\tau = 2\pi \int |g|^2(p)\delta(\omega(p)) \, dp \geq 0$. When $\gamma = 1$, the $A(\varphi,k)$ are smeared versions of the annihilators on $\Gamma(L^2(\mathbb{R}))$.

The limit $k \uparrow \infty$ corresponds to the smeared field becoming singular and this leads to a quantum Markovian approximation. The formulation of such models was first given and treated in a systematic way by Accardi, Frigero and Lu who developed a set of powerful quantum functional central limit theorems including the weak coupling [6] and low density [7] regimes. Theorem 2 is an extension of these which includes both quantum diffusion and jump terms [8, 9].

Theorem 2 *Let* $(\mathsf{E}_{\alpha\beta})$ *be bounded operators on a fixed separable Hilbert space* $\mathfrak{h}$ *labeled by* $\alpha, \beta \in \{0,1\}$ *with* $\mathsf{E}_{\alpha\beta}^\dagger = \mathsf{E}_{\beta\alpha}$ *and* $\|E_{11}\| < 2$. *Let*

$$\Upsilon(t,k) = \mathsf{E}_{11} \otimes a(t,k)^\dagger a(t,k) + \mathsf{E}_{10} \otimes a(t,k)^\dagger + \mathsf{E}_{01} \otimes a(t,k) + \mathsf{E}_{00} \otimes I$$

and

$$e(\varphi,k) = \exp\{A(\varphi,k) - A(\varphi,k)^\dagger\}\Omega_R$$

with Ω_R *the Fock vacuum of* $\mathcal{F}_R$. *The solution* $V(t,k)$ *to the equation*

$$\frac{d}{dt}V(t,k) = -i\Upsilon(t,k)V(t,k), \qquad V(0,k) = I,$$

exists and we have the limit

$$\lim_{k\to\infty}\langle u_1 \otimes e(\varphi,k)\big|V(t,k)\big|u_2 \otimes e(\psi,k)\rangle = \langle u_1 \otimes e(\varphi)\big|V(t)\big|u_2 \otimes e(\psi)\rangle$$

for all $u_1, u_2 \in \mathfrak{h}$ *and* $\varphi, \psi \in L^2(\mathbb{R})$, *where* V *is a unitary adapted process on* $\mathfrak{h} \otimes \Gamma(\mathbb{C}^n \otimes L^2(\mathbb{R}))$ *with coefficient matrix* $\mathbf{G}$ *given by*

$$\mathbf{G} = -i\mathbf{E} - i\frac{1}{2}\mathbf{G}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{E}, \tag{5}$$

where we assume $\int_{-\infty}^0 \rho(\tau)\,d\tau = \frac{1}{2}$.

The proof of the theorem is given in [8] and requires a development and a uniform estimation of the Dyson series expansion. Summability of the series requires that $\|E_{11}\| < 2$.

The triple $(\mathsf{S}, \mathsf{L}, \mathsf{H})$ from (4) obtained through (5) is

$$\begin{aligned} &\mathsf{S} = \frac{I - \frac{i}{2}\mathsf{E}_{11}}{I + \frac{i}{2}\mathsf{E}_{11}}, \qquad \mathsf{L} = -\frac{i}{I + \frac{i}{2}\mathsf{E}_{11}}\mathsf{E}_{10}, \\ &\mathsf{H} = \mathsf{E}_{00} + \mathsf{E}_{01}\,\mathrm{Im}\left\{\frac{1}{I + \frac{i}{2}\mathsf{E}_{11}}\right\}\mathsf{E}_{10}. \end{aligned} \tag{6}$$

Our objective is reappraise Theorem 2, where we will prove a related result by an alternative technique. Using the Trotter-Kato theorem, we will establish a stronger mode of convergence (uniformly on compact intervals of time and strongly in the Hilbert space) by means of a graph convergence of the Hamiltonians. The new approach has the advantage of been simpler and is likely to be more readily extended to other cases, for instance a continuum of input channels as originally treated in [1], which cannot be treated by the perturbative techniques used in the proof of Theorem 2.

2 Trotter-Kato theorems for quantum stochastic limits

Our main results will employ the Trotter-Kato theorem, which we recall next in a particularly convenient form. See [10], Theorem 3.17, or [11], Chapter VIII.7.

Theorem 3 (Trotter-Kato) *Let* $\mathcal{H}$ *be a Hilbert space and let* $U^{(k)}(\cdot)$ *and* $U(\cdot)$ *be strongly continuous one-parameter groups of unitaries on* $\mathcal{H}$ *with Stone generators* $H^{(k)}$ *and* H, *respectively. Let* $\mathcal{D}$ *be a core for* H. *The following are equivalent*

1. *For all* $f \in \mathcal{D}$ *there exist* $f^{(k)} \in \mathrm{Dom}(H^{(k)})$ *such that*

$$\lim_{k\to\infty} f^{(k)} = f, \qquad \lim_{k\to\infty} H^{(k)} f^{(k)} = Hf.$$

2. *For all* $0 \le T < \infty$ *and all* $f \in \mathcal{H}$ *we have*

$$\lim_{k\to\infty} \sup_{0\le t\le T} \left\| \left(U^{(k)}(t) - U(t)\right) f \right\| = 0.$$

The theorem yields a strong uniform convergence if we can establish graph convergence of the Hamiltonians. We now present the Trotter-Kato theorems for the class of problems that interest us, treating the first and second quantized problems in sequence.

2.1 First quantization example

Definition 4 Let $g \in C_c^\infty(\mathbb{R})$, i.e., an infinitely differentiable function with compact support, such that $\int_{-\infty}^{\infty} g(s)\,ds = 1$. We define $\rho(t) = \int_{\mathbb{R}} g(s)^* g(s+t)\,ds$. Moreover, for all $k > 0$, we define functions $g^{(k)}$ and $\rho^{(k)}$ by

$$g^{(k)}(t) = kg(kt), \qquad \rho^{(k)}(t) = k\rho(kt), \quad t \in \mathbb{R}.$$

Furthermore, we define two complex numbers by $\kappa_+ := \int_0^\infty \rho(s)\,ds$ and $\kappa_- := \int_{-\infty}^0 \rho(s)\,ds$.

Note that $\kappa_+ + \kappa_- = 1$ and that κ_+ and κ_- are complex conjugate: $\kappa_+ = (\kappa_-)^*$ (substitute $-s$ for s), hence $\kappa_\pm = \frac{1}{2} \pm i\sigma$ with σ real. The choice of ρ is such that $\langle g | g * f\rangle = \langle \rho | f\rangle$, where $(g * f)(t) = \int_{-\infty}^{\infty} g(s) f(t-s)\,ds$ is the usual convolution.

Let $\mathfrak{h}$ be a Hilbert space and let E be a bounded self-adjoint operator on $\mathfrak{h}$. We consider the following family of operators on $L^2(\mathbb{R}; \mathfrak{h}) \simeq \mathfrak{h} \otimes L^2(\mathbb{R})$:

$$\begin{aligned} &H^{(k)} = i\partial + \mathsf{E}\left|g^{(k)}\right\rangle\!\left\langle g^{(k)}\right| \simeq I \otimes i\partial + E \otimes \left|g^{(k)}\right\rangle\!\left\langle g^{(k)}\right|, \\ &\mathrm{Dom}\left(H^{(k)}\right) = W^{1,2}(\mathbb{R}; \mathfrak{h}), \end{aligned} \tag{7}$$

where $W^{1,2}(X; \mathfrak{h})$, $X \subseteq \mathbb{R}$, denotes the Sobolev space of $\mathfrak{h}$-valued functions square integrable on X with square integrable weak derivatives on X. It follows easily that $H^{(k)}$ is self-adjoint for every $k > 0$ (for example by the Kato-Rellich theorem, see [12], Theorem X.12). We define a unitary operator on $\mathfrak{h}$ by

$$\mathsf{S} = \frac{I - i\kappa_- \mathsf{E}}{I + i\kappa_+ \mathsf{E}} \tag{8}$$

and an operator H on $L^2(\mathbb{R}; \mathfrak{h})$ by

$$\begin{aligned} &\mathrm{Dom}(H) = \left\{ f \in W^{1,2}\left(\mathbb{R}\backslash\{0\}; \mathfrak{h}\right) : f\left(0^-\right) = \mathsf{S} f\left(0^+\right) \right\}, \\ &Hf = i\partial f. \end{aligned} \tag{9}$$

It follows easily that H is self-adjoint, compare [11], VIII.2, final example.

Remark Any $f \in W^{1,2}(\mathbb{R}\backslash\{0\}; \mathfrak{h})$ is absolutely continuous both on $(-\infty, 0)$ and $(0, \infty)$, see for example [13], 2.6 Ex. 6, but the exclusion of test functions supported at 0 allows jumps at 0. Higher dimensional situations ($\mathbb{R}^n$ with $n > 1$) are more complicated in this respect.

We define strongly continuous one-parameter groups of unitaries on $L^2(\mathbb{R};\mathfrak{h})$ by

$$U^{(k)}(t) = \exp\bigl(-itH^{(k)}\bigr), \qquad U(t) = \exp(-itH).$$

We then have the following theorem.

Theorem 5 *Let* $0 \leq T < \infty$. *Then*

$$\lim_{k\to\infty} \sup_{0\leq t\leq T} \bigl\| \bigl(U^{(k)}(t) - U(t)\bigr)f \bigr\| = 0, \quad \forall f \in L^2(\mathbb{R};\mathfrak{h}).$$

We prove Theorem 5 at the end of this subsection. From the Trotter-Kato Theorem 3, it suffices to find, for every $f \in \mathrm{Dom}(H)$, a sequence $f^{(k)} \in \mathrm{Dom}(H^{(k)})$ that satisfies condition (i) of Theorem 3.

If g is a $\mathbb{C}$-valued function on X and $f \in L^2(X;\mathfrak{h}) \simeq \mathfrak{h} \otimes L^2(X;\mathbb{C})$ then we use the short notation gf for $(I \otimes M_g)f$ where M_g is multiplication by g. With this convention we can also define $g * f \in L^2(X;\mathfrak{h})$ and $\langle g|f\rangle \in \mathfrak{h}$ for suitable functions g, using the same formulas as for $\mathfrak{h} = \mathbb{C}$.

Definition 6 Let f be an element in the domain of H. Define an element $f^{(k)}$ in the domain of $H^{(k)}$ by

$$f^{(k)}(t) = \bigl(g^{(k)} * f\bigr)(t) = \int_{-\infty}^{\infty} g^{(k)}(t-s) f(s)\, ds.$$

Lemma 7 *Let* η *be an element of* $C(0,\infty)$ *with compact support and let* h *be an element of* $W^{1,2}((0,\infty);\mathfrak{h}) \cap C^1((0,\infty);\mathfrak{h})$ *such that* $h(0^+) = 0$. *Let* $\eta^{(k)}(x) = k\eta(kx)$ *for all* $x \in (0,\infty)$ *and* $k > 0$. *Then*

$$\bigl\|\langle \eta^{(k)}|h\rangle\bigr\|_2 \leq \frac{C}{k}, \quad \forall k > 0,$$

for some positive constant C.

Proof Note that the C^1-function h is Lipschitz on the support of η, that is, there exists a positive constant L such that

$$\bigl\|h(x) - h(y)\bigr\|_2 \leq L|x-y|, \quad \forall x,y \in \mathrm{supp}(\eta),$$

where $\mathrm{supp}(\eta)$ denotes the support of η. Taking the limit for y to 0^+ gives

$$\bigl\|h(x)\bigr\|_2 \leq L|x|, \quad x \in \mathrm{supp}(\eta).$$

We can define $M := \max_{x\in(0,\infty)} |\eta(x)|$ and let N be a number to the right of the support of η. Now we have

$$\begin{aligned} \bigl\|\langle \eta^{(k)}|h\rangle\bigr\|_2 &\leq k \int_0^{\infty} \bigl|\eta(kx)\bigr| \bigl\|h(x)\bigr\|_2\, dx \\ &\leq \frac{L}{k} \int_0^{\infty} \bigl|\eta(u)\bigr| u\, du \leq \frac{L}{k} \int_0^{N} Mu\, du = \frac{LMN^2}{2k}. \end{aligned}$$

□

Lemma 8 *If f is in* $\mathrm{Dom}(H) \cap C^\infty(\mathbb{R}\backslash\{0\}; \mathfrak{h})$, *and $f^{(k)}$ is given by Definition* 6, *then we have*

$$1. \quad \lim_{k\to\infty} \|f^{(k)} - f\|_2 = 0, \qquad 2. \quad \lim_{k\to\infty} \|H^{(k)} f^{(k)} - Hf\|_2 = 0.$$

Proof Note that the first limit follows immediately from a standard result on approximations by convolutions, see e.g. [14], Thm. 2.16. For the second limit, note that

$$\partial\big(g^{(k)} * f\big) = g^{(k)} * \partial f + \big(f(0^+) - f(0^-)\big)g^{(k)}, \tag{10}$$

because $\partial f = Hf$ and using [14], Thm. 2.16, once more, we find that

$$\lim_{k\to\infty} g^{(k)} * Hf = Hf.$$

That is, all we need to show is that

$$\lim_{k\to\infty} \big\| \big(if(0^+) - if(0^-) + \mathsf{E}\langle g^{(k)}|g^{(k)} * f\rangle\big) g^{(k)} \big\|_2 = 0. \tag{11}$$

Note that $\langle g^{(k)}|g^{(k)} * f\rangle = \langle \rho^{(k)}|f\rangle$. We can now apply Lemma 7 with $h = f\chi_{(0,\infty)} - f(0^+)$ and $\eta = \rho\chi_{(0,\infty)}$ (resp. $h = f\chi_{(-\infty,0)} - f(0^-)$ and $\eta = \rho\chi_{(-\infty,0)}$) to conclude that

$$\langle \rho^{(k)}|f\rangle \overset{k\to\infty}{\longrightarrow} (\kappa_-)^* f(0^-) + (\kappa_+)^* f(0^+) = \kappa_+ f(0^-) + \kappa_- f(0^+),$$

with rate $\frac{1}{k}$. Using the boundary condition for f, we therefore find that

$$if(0^+) - if(0^-) + \mathsf{E}\langle g^{(k)}|g^{(k)} * f\rangle \longrightarrow i\big[(I - i\kappa_-\mathsf{E})f(0^+) - (I + i\kappa_+\mathsf{E})f(0^-)\big] = 0,$$

with rate $\frac{1}{k}$. Note that the L^2-norm of $g^{(k)}$ grows with rate $\sqrt{k}$, so that the limit in equation (11) follows. This completes the proof of the lemma. □

Proof of Theorem 5 The theorem follows from a combination of the results in Theorem 3 and Lemma 8 and the fact that $\mathrm{Dom}(H) \cap C^\infty(\mathbb{R}\backslash\{0\}; \mathfrak{h})$ is a core for H. The latter follows from [14], Thm. 7.6. □

3 A second quantized model

Let $\mathsf{E}_{\alpha\beta}$ be bounded operators on $\mathfrak{h}$ such that $\mathsf{E}^\dagger_{\alpha\beta} = \mathsf{E}_{\beta\alpha}$ for $\alpha, \beta \in \{0,1\}$. Consider the following family of operators on $\mathfrak{h} \otimes \mathcal{F}$

$$H^{(k)} = i\, d\Gamma(\partial) + \mathsf{E}_{11} A^\dagger\big(g^{(k)}\big) A\big(g^{(k)}\big) + \mathsf{E}_{10} A^\dagger\big(g^{(k)}\big) + \mathsf{E}_{01} A\big(g^{(k)}\big) + \mathsf{E}_{00}, \tag{12}$$

choosing a suitable domain $\mathrm{Dom}(H^{(k)})$ of essential self-adjointness for all $k > 0$. (We conjecture that $\mathfrak{h} \otimes \mathcal{E}(C_c^\infty(\mathbb{R}))$, where $\mathcal{E}(C_c^\infty(\mathbb{R}))$ is the set of exponential vectors $e(f)$ with $f \in C_c^\infty(\mathbb{R})$, is a set of analytic vectors for the $H^{(k)}$ but we haven't been able to prove this rigorously and leave it as an open problem.)

We denote the strongly continuous group of unitaries on $\mathfrak{h} \otimes \mathcal{F}$ generated by the unique self-adjoint extension of $H^{(k)}$ by $U^{(k)}(t)$. Let the triple $(\mathsf{S}, \mathsf{L}, \mathsf{H})$ appearing in (4) be obtained from $\mathbf{E} = (\mathsf{E}_{\alpha\beta})$ through (5): see (6).

The space $\mathfrak{h} \otimes \mathcal{F} = \mathfrak{h} \otimes \Gamma(L^2(\mathbb{R}))$ consists of vectors $\Psi = (\Psi_m)_{m\geq 0}$ which are sequences of symmetric $\mathfrak{h}$-valued functions $\Psi_m(t_1, \ldots, t_m)$ where $t_j \in \mathbb{R}$. Following Gregoratti [2], we define the following spaces (for I a Borel subset of $\mathbb{R}$ and $\mathfrak{H}$ a Hilbert space):

$$\mathcal{H}^{\Sigma}\big(I^m, \mathfrak{H}\big) = \left\{ v \in L^2\big(I^m, \mathfrak{H}\big) : \sum_{i=1}^{m} \partial_i v \in L^2\big(I^m, \mathfrak{H}\big) \right\};$$

$$\mathcal{W} = \left\{ \Psi \in \mathfrak{h} \otimes \mathcal{F} : \Psi_m \in \mathcal{H}^{\Sigma}\big(\mathbb{R}^m, \mathfrak{h}\big) : \sum_{m=0}^{\infty} \frac{1}{m!} \left\| \sum_{i=1}^{m} \partial_i \Psi_m \right\|^2 < \infty \right\};$$

$$\mathcal{V}_s = \left\{ \Psi \in \mathcal{W} : \sum_{m=0}^{\infty} \frac{1}{m!} \big\| \Psi_{m+1}(\cdot, t_{m+1} = s) \big\|^2 < \infty \right\};$$

$$\mathcal{V}_{0^\pm} = \mathcal{V}_{0^+} \cap \mathcal{V}_{0-}.$$

We remark that $\mathcal{W}$ is the natural domain for $d\Gamma(i\partial)$. On $\mathcal{V}_s$ we define the operators

$$\big(a(s)\Psi\big) = \Psi_{n+1}(\cdot, t_{n+1} = s).$$

On the subspace $\mathcal{V}_{0^\pm}$, the operators $d\Gamma(i\partial)$ and $a(0^\pm)$ are all simultaneously defined.

Definition 9 (The Gregoratti Hamiltonian) Define the following operator H on $\mathfrak{h} \otimes \mathcal{F}$

$$H\Phi = d\Gamma(i\partial_{ac})\Phi - i\mathsf{L}^\dagger \mathsf{S} a\big(0^+\big)\Phi + \left(\mathsf{H} - \frac{i}{2}\mathsf{L}^\dagger\mathsf{L}\right)\Phi, \tag{13}$$

$$\mathrm{Dom}(H) = \big\{\Phi \in \mathcal{V}_{0^\pm} : a\big(0^-\big)\Phi = \mathsf{S} a\big(0^+\big)\Phi + \mathsf{L}\Phi\big\}. \tag{14}$$

It follows from the work of Chebotarev and Gregoratti [2, 4] that the operator H is essentially self-adjoint and its unique self-adjoint extension generates the unitary group $U(t) = \Theta_t V_t$ where V_t is the unitary solution to the following quantum stochastic differential equation (3):

$$\begin{aligned} dV(t) &= \left\{ (\mathsf{S} - 1)\, d\Lambda(t) + \mathsf{L}\, dA^\dagger(t) - \mathsf{L}^\dagger \mathsf{S}\, dA(t) - \frac{1}{2}\mathsf{L}^\dagger \mathsf{L}\, dt - i\mathsf{H}\, dt \right\} V(t), \\ V(0) &= I. \end{aligned} \tag{15}$$

The main result of this section is the following theorem.

Theorem 10 *Let* $0 \leq T < \infty$. *We have the following*

$$\lim_{k\to\infty} \sup_{0\leq t\leq T} \big\| \big(U^{(k)}(t) - U(t)\big)\Phi \big\| = 0, \quad \forall \Phi \in \mathfrak{h} \otimes \mathcal{F}.$$

Before proving the theorem (see the end of this section), we make some preparations. As in the previous section, we would like to use the Trotter-Kato theorem, therefore, for every Φ in a core for $\mathrm{Dom}(H)$, we need to construct an approximating sequence $\Phi^{(k)}$ that satisfies the first condition of Theorem 3. We again employ a smearing through convolution with $g^{(k)}$, this time applied as a second quantization.

Definition 11 Let $g^{(k)}$ be as in Definition 4 and assume further that $g(t) \geq 0$ for all t (hence $\|g\|_1 = 1$). Let $G^{(k)} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the convolution with $g^{(k)}$, i.e.

$$G^{(k)}h = g^{(k)} * h, \quad \forall h \in L^2(\mathbb{R}).$$

Let Φ be an element in $\mathrm{Dom}(H)$. We define an element $\Phi^{(k)}$ in the domain of $H^{(k)}$ by

$$\Phi^{(k)} = \Gamma\bigl(G^{(k)}\bigr)\Phi. \tag{16}$$

Here $\Gamma(G^{(k)})$ denotes the second quantization of $G^{(k)}$.

Note that $G^{(k)}$ is a contraction ($\|g^{(k)}\|_1 = 1$, i.e. $\|\hat{g}^{(k)}\|_\infty \leq 1$ with $\hat{g}^{(k)}$ the Fourier transform $\hat{g}^{(k)} = \int_{-\infty}^{\infty} g^{(k)}(t)e^{-i\omega t}\,dt$), so its second quantization is well-defined). The positivity assumption on g implies that $\kappa_+ = \kappa_- = \frac{1}{2}$ (which agrees with Section 1.2).

Lemma 12 *For all $\Phi \in \mathfrak{h} \otimes \mathcal{F}$, we have*

$$\lim_{k\to\infty} \Gamma\bigl(G^{(k)}\bigr)\Phi = \Phi.$$

Proof Since the linear span of exponential vectors $v \otimes e(h)$ is dense in $\mathfrak{h} \otimes \mathcal{F}$ and $\Gamma(G^{(k)})$ is bounded, it is enough to prove the lemma for all vectors of the form $\Phi = v \otimes e(h)$. We have

$$\begin{aligned}&\bigl\|\Gamma\bigl(G^{(k)}\bigr)v \otimes e(h) - v \otimes e(h)\bigr\|^2 \\ &\quad = \|v\|^2\bigl[\exp\bigl(\bigl\|G^{(k)}h\bigr\|^2\bigr) + \exp\bigl(\|h\|^2\bigr) - \exp\bigl(\bigl\langle G^{(k)}h|h\bigr\rangle\bigr) - \exp\bigl(\bigl\langle h|G^{(k)}h\bigr\rangle\bigr)\bigr] \to 0,\end{aligned}$$

where in the last step we used [14], Thm. 2.16. □

We now recall the following result, see for instance [15].

Lemma 13 *Let $C : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be a contraction. We have for $h \in L^2(\mathbb{R})$*

$$\Gamma(C)\bigl(\mathrm{Dom}\bigl(A\bigl(C^\dagger h\bigr)\bigr)\bigr) \subset \mathrm{Dom}\bigl(A(h)\bigr).$$

Moreover, on the domain of $A(C^\dagger h)$, we have

$$A(h)\Gamma(C) = \Gamma(C)A\bigl(C^\dagger h\bigr).$$

Note that we have the following second quantized version of equation (10):

$$d\Gamma(i\partial)\Phi^{(k)} = \Gamma\bigl(G^{(k)}\bigr)\,d\Gamma(i\partial_{ac})\Phi + iA^\dagger\bigl(g^{(k)}\bigr)\Gamma\bigl(G^{(k)}\bigr)a_J\Phi,$$

where

$$(a_J\Phi)_m(t_1,\dots,t_m) = \Phi_{m+1}\bigl(t_1,\dots,t_m,0^+\bigr) - \Phi_{m+1}\bigl(t_1,\dots,t_m,0^-\bigr).$$

The action of $H^{(k)}$ on $\Phi^{(k)}$ can now be written as

$$H^{(k)}\Phi^{(k)} = \Gamma\big(G^{(k)}\big)\,d\Gamma(i\partial_{ac})\Phi + A^{\dagger}\big(g^{(k)}\big)\Gamma\big(G^{(k)}\big)\big(ia_{J}\Phi + \mathsf{E}_{11}A\big(\rho^{(k)}\big)\Phi + \mathsf{E}_{10}\Phi\big) \\ + \mathsf{E}_{01}\Gamma\big(G^{(k)}\big)A\big(\rho^{(k)}\big)\Phi + \mathsf{E}_{00}\Gamma\big(G^{(k)}\big)\Phi. \tag{17}$$

Here we have used Lemma 13 and the fact that $A(G^{(k)\dagger}g^{(k)}) = A(\rho^{(k)})$.

Lemma 14 *The singular component of equation* (17) *converges strongly to zero as $k \to \infty$, i.e.*,

$$\big\|A^{\dagger}\big(g^{(k)}\big)\Gamma\big(G^{(k)}\big)\big(ia_{J}\Phi + \mathsf{E}_{11}A\big(\rho^{(k)}\big)\Phi + \mathsf{E}_{10}\Phi\big)\big\|_{2} \stackrel{k\to\infty}{\longrightarrow} 0,$$

for all Φ in a core domain $\mathcal{D}$ of H.

We defer the proof of this lemma to the next section.

Using Lemma 12, we find that the first term in equation (17) converges to the first term in the Hamiltonian H given by equation (13), i.e.

$$\lim_{k\to\infty}\big\|\Gamma\big(G^{(k)}\big)\,d\Gamma(i\partial_{ac})\Phi - d\Gamma(i\partial_{ac})\Phi\big\|_{2} = 0.$$

In the proof of the Lemma 14, it is shown that $A(\rho^{(k)})\Phi$ converges in L^2-norm to $\frac{1}{2}a(0^-)\Phi + \frac{1}{2}a(0^+)\Phi$. Therefore, we find for the last line of equation (17)

$$\mathsf{E}_{01}\Gamma\big(G^{(k)}\big)A\big(\rho^{(k)}\big)\Phi + \mathsf{E}_{00}\Gamma\big(G^{(k)}\big)\Phi \longrightarrow \mathsf{E}_{01}\left(\frac{1}{2}a\big(0^{+}\big) + \frac{1}{2}a\big(0^{-}\big)\right)\Phi + \mathsf{E}_{00}\Phi.$$

Employing the boundary condition, we have that

$$\begin{aligned} &\mathsf{E}_{01}\left(\frac{1}{2}a\big(0^{+}\big) + \frac{1}{2}a\big(0^{-}\big)\right)\Phi + \mathsf{E}_{00}\Phi \\ &\quad = \mathsf{E}_{01}\left(\frac{1}{2}a\big(0^{+}\big)\Phi + \frac{1}{2}\big[\mathsf{S}a\big(0^{+}\big)\Phi + \mathsf{L}\Phi\big]\right) + \mathsf{E}_{00}\Phi \\ &\quad \equiv -i\mathsf{L}^{\dagger}\mathsf{S}a\big(0^{+}\big)\Phi + \left(\mathsf{H} - \frac{i}{2}\mathsf{L}^{\dagger}\mathsf{L}\right)\Phi. \end{aligned}$$

Here we have used the algebraic identities

$$\mathsf{E}_{01}\left(\frac{1}{2} + \frac{1}{2}\mathsf{S}\right) = \mathsf{E}_{01}\left(\frac{1}{2} + \frac{1}{2}\frac{I - i\frac{1}{2}\mathsf{E}_{11}}{I + i\frac{1}{2}\mathsf{E}_{11}}\right) = \mathsf{E}_{01}\frac{1}{I + i\frac{1}{2}\mathsf{E}_{11}} \equiv -iL^{\dagger}\mathsf{S},$$

$$-i\frac{\frac{1}{2}}{I + i\frac{1}{2}\mathsf{E}_{11}} = \frac{1}{2}\,\mathrm{Im}\left\{\frac{\frac{1}{2}}{I + i\frac{1}{2}\mathsf{E}_{11}}\right\} - \frac{i}{2}\frac{I}{I + i\frac{1}{2}\mathsf{E}_{11}}\frac{I}{I - i\frac{1}{2}\mathsf{E}_{11}}.$$

Applying the Trotter-Kato theorem, this completes the proof of our main result Theorem 10.

4 Proof of Lemma 14

Setting $V^{(k)} = ia_J\Phi + \mathsf{E}_{11}A(\rho^{(k)})\Phi + \mathsf{E}_{10}\Phi$, we see that

$$\begin{aligned}&\|A^\dagger(g^{(k)})\Gamma(G^{(k)})V^{(k)}\|_2^2\\&\quad=\langle\Gamma(G^{(k)})V^{(k)}|A(g^{(k)})A^\dagger(g^{(k)})\Gamma(G^{(k)})V^{(k)}\rangle\\&\quad=\langle\Gamma(G^{(k)})V^{(k)}|(A^\dagger(g^{(k)})A(g^{(k)})+\|g^{(k)}\|_2^2)\Gamma(G^{(k)})V^{(k)}\rangle\\&\quad\le\|A(g^{(k)})\Gamma(G^{(k)})V^{(k)}\|_2^2+\|g^{(k)}\|_2^2\|V^{(k)}\|_2^2,\end{aligned}$$

where in the last step we used that $\Gamma(G^{(k)})$ is a contraction. We need to establish two further results: the first is that $V^{(k)}$ goes to 0 sufficiently quickly and we prove this in Lemma 16 below; then we will have to show that this implies that the first term $\|A(g^{(k)})\Gamma(G^{(k)})V^{(k)}\|_2^2$ converges to 0 and we prove this in Lemma 17.

If we accept these results for the moment, then from the boundary conditions we have

$$\begin{aligned}&ia_J\Phi+\mathsf{E}_{11}\left(\frac{1}{2}a(0^+)+\frac{1}{2}a(0^-)\right)\Phi+\mathsf{E}_{10}\Phi\\&\quad=i\left(I-i\frac{1}{2}\mathsf{E}_{11}\right)a(0^+)\Phi-i\left(I+i\frac{1}{2}\mathsf{E}_{11}\right)a(0^-)\Phi+\mathsf{E}_{10}\Phi\\&\quad=i\left(I+i\frac{1}{2}\mathsf{E}_{11}\right)\left[Sa(0^+)\Phi+L\Phi-a(0^-)\Phi\right]=0\end{aligned}$$

so that, in fact,

$$V^{(k)}=\mathsf{E}_{11}\left[A(\rho^{(k)})\Phi-\left(\frac{1}{2}a(0^+)+\frac{1}{2}a(0^-)\right)\Phi\right].$$

As $\|g^k\|_2$ grows at rate $\sqrt{k}$, it suffices to show that $A(\rho^{(k)})\Phi-(\frac{1}{2}a(0^+)+\frac{1}{2}a(0^-))\Phi$ goes to 0 in norm with rate faster than $\frac{1}{\sqrt{k}}$. We will now establish this result below, but first we need to recall the definition of a pseudo-exponential vector from [2].

Definition 15 Let $\mathsf{F}: t\mapsto\mathsf{F}_t$ be a function from $\mathbb{R}$ to $\mathfrak{B}(\mathfrak{h})$ and define the corresponding pseudo-exponential vector $\Psi(\mathsf{F},h)$ as

$$\left[\Psi(\mathsf{F},h)\right]_m(t_1,\ldots,t_m)=\vec{T}\mathsf{F}_{t_1}\cdots\mathsf{F}_{t_m}h$$

for given $h\in\mathfrak{h}$, where $\vec{T}$ denotes chronological ordering. That is

$$\vec{T}\mathsf{F}_{t_1}\cdots\mathsf{F}_{t_m}=\mathsf{F}_{t_{\sigma(1)}}\cdots\mathsf{F}_{t_{\sigma(m)}},$$

where σ is a permutation for which $t_{\sigma(1)}\ge\cdots\ge t_{\sigma(m)}$.

Lemma 16 *Let $v\in W^{1,2}(\mathbb{R}/\{0\})$ and $u\in W^{1,2}(\mathbb{R}/\{0\})$ with $u|_{\mathbb{R}_+}=0$ and $u(0^-)=1$, then define F_t by*

$$\mathsf{F}_t=v(t)+u(t)\left[\mathsf{S}v(0^+)+\mathsf{L}-v(0^-)\right]\tag{18}$$

then the domain $\mathcal{D}$ of such pseudo-exponential vectors $\Phi = \Psi(\mathsf{F}, h)$ is a core for H. Moreover, for each such vector we have

$$\left\| A\bigl(\rho^{(k)}\bigr)\Phi - \left(\frac{1}{2}a\bigl(0^+\bigr) + \frac{1}{2}a\bigl(0^-\bigr)\right)\Phi \right\|_2 = O\left(\frac{1}{k}\right).$$

Proof The first part of this lemma is proved by Gregoratti where it is shown that $\mathcal{D}$ is dense, and is contained in $\mathrm{Dom}(H) \cap \mathcal{V}_{0\pm}$, see [2], Propositions 4 and 5. Note that for $\Phi = \Psi(\mathsf{F}, h)$, by (4) in [2] we have

$$a(t)\Phi = \nu(t)\Phi, \quad t \in \bigl\{0^+\bigr\} \cup (0, \infty),$$
$$a\bigl(0^-\bigr)\Phi = \bigl(\mathsf{S}\nu\bigl(0^+\bigr) + \mathsf{L}\bigr)\Phi.$$

To prove the second part, we begin by setting

$$\begin{aligned} Z_m(t_1, \dots, t_m) &= \left[A\bigl(\rho^{(k)}\bigr)\Phi - \left(\frac{1}{2}a\bigl(0^+\bigr) + \frac{1}{2}a\bigl(0^-\bigr)\right)\Phi\right]_m (t_1, \dots, t_m) \\ &= \int_0^\infty \rho^{(k)}(s)\bigl[\Phi_{m+1}(t_1, \dots, t_m, s) - \Phi_{m+1}\bigl(t_1, \dots, t_m, 0^+\bigr)\bigr]\, ds \\ &\quad + \int_{-\infty}^0 \rho^{(k)}(s)\bigl[\Phi_{m+1}(t_1, \dots, t_m, s) - \Phi_{m+1}\bigl(t_1, \dots, t_m, 0^-\bigr)\bigr]\, ds \\ &\equiv Z_m^+(t_1, \dots, t_m) + Z_m^-(t_1, \dots, t_m). \end{aligned}$$

We have $\|Z_m\|^2 \le (\|Z_m^+\| + \|Z_m^-\|)^2$ but

$$Z_m^+(t_1, \dots, t_m) = \int_0^\infty \rho^{(k)}(s)\bigl[\nu(s) - \nu\bigl(0^+\bigr)\bigr]\, ds\Phi_m(t_1, \dots, t_m)$$

and this prefactor is clearly $O(\frac{1}{k})$ from the argument used in Lemma 8.

However, we then have

$$\begin{aligned} &Z_m^-(t_1, \dots, t_m) \\ &\quad = \int_{-\infty}^0 \rho^{(k)}(s)[\mathsf{F}_{t_{\sigma(1)}} \cdots \mathsf{F}_s \cdots \mathsf{F}_{t_{\sigma(m)}} - \mathsf{F}_{0^-}\mathsf{F}_{t_{\sigma(1)}} \cdots \mathsf{F}_{t_{\sigma(m)}}]h\, ds, \end{aligned}$$

where σ is the chronological time ordering permutation.

We note however that $[\mathsf{F}_t, \mathsf{F}_s] = 0$ for all t, s, therefore we have

$$\begin{aligned} Z_m^-(t_1, \dots, t_m) &= \int_{-\infty}^0 \rho^{(k)}(s)[\mathsf{F}_s - \mathsf{F}_{0^-}]\mathsf{F}_{t_{\sigma(1)}} \cdots \mathsf{F}_{t_{\sigma(m)}}h\, ds \\ &= \int_{-\infty}^0 \rho^{(k)}(s)\bigl[u(s) - u\bigl(0^-\bigr)\bigr]\bigl[\mathsf{S}\nu\bigl(0^+\bigr) + \mathsf{L} - \nu\bigl(0^-\bigr)\bigr] \\ &\quad \times \mathsf{F}_{t_{\sigma(1)}} \cdots \mathsf{F}_{t_{\sigma(m)}}h\, ds, \end{aligned}$$

where we used (18). From the argument in Lemma 8 again, we see that this is $O(\frac{1}{k})$. □

Lemma 17 *For Φ chosen as a pseudo-exponential vector, as in Lemma* 16, *we have that* $\|A(g^{(k)})\Gamma(G^{(k)})V^{(k)}\|_2^2$ *converges to* 0 *as* $k \to \infty$.

Proof We have that

$$A(g^{(k)})\Gamma(G^{(k)})V^{(k)} = \Gamma(G^{(k)})A(\rho^{(k)})V^{(k)},$$

with $\Gamma(G^{(k)})$ a contraction. The mth level of the Fock space component of $A(\rho^{(k)})V^{(k)}$ may be written as

$$E_{11}A(\rho^{(k)})Z_m^+ + E_{11}A(\rho^{(k)})Z_m^-,$$

where we use the same conventions as in Lemma 16. The first term has the explicit components

$$\begin{aligned} &E_{11}\int dt\rho^{(k)}(t)\int_0^\infty \rho^{(k)}(s)\big[\nu(s)-\nu(0^+)\big]\,ds\Phi_{m+1}(t,t_1,\ldots,t_m) \\ &\quad = E_{11}\int dt\rho^{(k)}(t)\mathsf{F}_t\int_0^\infty \rho^{(k)}(s)\big[\nu(s)-\nu(0^+)\big]\,ds\Phi_m(t_1,\ldots,t_m) \end{aligned}$$

which is norm bounded by $\|E_{11}\|\int dt\rho^{(k)}(t)\|\mathsf{F}_t\|\|Z_m^+\|$, and we note that in fact $\int dt\rho^{(k)}(t)\times\|\mathsf{F}_t\| = \int d\tau\rho(\tau)\|\mathsf{F}_{\tau/k}\|$. An equivalent bound is easily shown to hold for $E_{11}A(\rho^{(k)})Z_m^-$ and so by an argument similar to lemma 16 we obtain the desired result. □

4.1 Epilogue

After completion of this work, the authors became aware of the book by W. von Waldenfels [16] which gives a complete resolvent analysis of the Chebotarev-Gregoratti-von Waldenfels Hamiltonian, and in the final chapter describes a strong resolvent limit by colored noise approximations. The convergence is comparable to the strong uniform convergence considered here, but the approach is very different.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]BTA VOF, Slootsekuilen 9, 5986 PE, Beringe, The Netherlands. [2]Dept. for Mathematics and Physics, Aberystwyth University, SY23 3BZ, Wales, United Kingdom. [3]School of Electrical Engineering and Telecommunications, UNSW Australia, Sydney, NSW 2052, Australia.

Acknowledgements
JG and RG are grateful to EPSRC EP/G039275/1 and EP/L006111/1 grant for support. They also wish to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme *Quantum Control Engineering* where work on this paper was completed. HN acknowledges support through a research visit funded through EPSRC EP/H016708/1 and Australian Research Council grants DP0986615 and DP130104191.

Received: 24 September 2014 Accepted: 14 April 2015 Published online: 30 April 2015

References

1. Hudson RL, Parthasarathy KR. Quantum Ito's formula and stochastic evolutions. Commun Math Phys. 1984;93:301-23.
2. Gregoratti M. The Hamiltonian operator associated to some quantum stochastic differential equations. Commun Math Phys. 2001;222:181-200.

3. von Waldenfels W. Symmetric differentiation and Hamiltonian of a quantum stochastic process. Infin Dimens Anal Quantum Probab Relat Top. 2005;8(1):73-116.
4. Chebotarev AM. Quantum stochastic differential equation is unitarily equivalent to a symmetric boundary problem for the Schrödinger equation. Math Notes. 1997;61(4):510-8.
5. Quezada-Batalla R, González-Gaxiola O. On the Hamiltonian of a class of quantum stochastic processes. Math Notes. 2007;81(5-6):734-52.
6. Accardi L, Frigerio A, Lu YG. The weak coupling limit as a quantum functional central limit. Commun Math Phys. 1990;131:537-70.
7. Accardi L, Frigerio A, Lu YG. The low density limit in finite temperature case. Nagoya Math J. 1992;126:25-87.
8. Gough J. Quantum flows as Markovian limit of emission, absorption and scattering interactions. Commun Math Phys. 2005;254:489-512.
9. Gough J. Quantum Stratonovich calculus and the quantum Wong-Zakai theorem. J Math Phys. 2006;47:113509.
10. Davies EB. One-parameter semigroups. London: Academic Press; 1980.
11. Reed M, Simon B. Methods of mathematical physics I: functional analysis. New York: Academic Press; 1980.
12. Reed M, Simon B. Methods of mathematical physics II: Fourier analysis, self-adjointness. New York: Academic Press; 1975.
13. Gustafson KE. Introduction to partial differential equations and Hilbert space methods. 3rd ed. New York: Dover; 1999.
14. Lieb EH, Loss M. Analysis. Providence: Am. Math. Soc.; 1997.
15. Petz D. Invitation to the canonical commutation relations. Leuven: Leuven University Press; 1990.
16. von Waldenfels W. A measure theoretical approach to quantum stochastic processes. Lecture notes in physics. vol. 878. Berlin: Springer; 2014.

Submit your manuscript to a SpringerOpen journal and benefit from:

- ► Convenient online submission
- ► Rigorous peer review
- ► Immediate publication on acceptance
- ► Open access: articles freely available online
- ► High visibility within the field
- ► Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com